Anti-plane problem of a lip-shape crack in one-dimensional hexagonal quasi-crystal materials

Jing Yu[1,2], JunHong Guo[1,*], YongMing Xing[1]

[1] College of Science, Inner Mongolia University of Technology, Hohhot 010051, China
[2] College of general education, Inner Mongolia Normal University, Hohhot 011517, China
* Corresponding author: jhguo@imut.edu.cn

Abstract By introducing a conformal mapping and using the complex variable function method, the fracture behavior of a lip-shape crack in one-dimensional hexagonal quasi-crystals materials is investigated under anti-plane loading at infinity. The expressions for stress, strains, displacements and field intensity factors of the phonon and the phason fields in the vicinity of the crack tip are obtained. When the height of the lip-shape crack approaches to zero, the present results can be reduced to the solutions of the Griffith crack

Keywords one-dimensional hexagonal quasi-crystals; lip-shape crack; complex variable function method; stress intensity factor

1. Introduction

The discovery of quasi-crystals (QCs) in 1984 is a significant breakthrough for condensed matter physics, which won the Nobel's award in 2011 [1]. A theoretical description of the deformed state of QCs requires a combined consideration of interrelated phonon and phason fields. The phonon field describes the motion of lattices in physical space, while the phason field describes quasiperiodic arrangement of atoms in the complementary orthogonal space, which interact with one another. Since the discovery of QCs, they have attracted the extensive attention of researchers engaged in experimental and theoretical work. A quantity of significant achievements of QCs have been done [2-11] recent years. Experiments have shown that quasi-crystals are quite brittle and the defects of quasicrystalline materials have been observed [12,13]. When quasicrystalline materials are subjected to mechanical stresses in service, the propagation of flaws or defects produced during their manufacturing process may result in premature failure of these materials. Therefore, the study of crack problem of quasicrystalline materials is meaningful both in theoretical and practical applications.

At present, the study on the fracture problems of quasicrystalline materials is mainly confined to relatively simple defects. Thus, the elastic problem of one-dimensional (1D) hexagonal quasicrystal materials becomes the primary object and made many of significant achievements. A moving screw dislocation in 1D hexagonal QCs was investigated [14]. The exact solutions of a semi-infinite crack and two semi-infinite collinear cracks in a strip of 1D hexagonal QCs were obtained [15,16]. The interaction of between dislocations and cracks in 1D hexagonal QCs were considered by the complex variable function method. Very recently, the analytical solutions of several complicated defects such as cracks originating from holes in 1D hexagonal QCs were obtained [17-19].

In this paper, by using the Stroh-type formulism for anti-plane deformation in 1D hexagonal QCs, the fracture mechanic of a lip-shape crack in a 1D hexagonal QC is investigated under uniform remote anti-plane shear loadings of the phonon field and the phason field. By introducing a conformal mapping and using the complex variable function method, which is further solved analytically. The expressions for stress, strains, displacements and field intensity factors of the phonon and the phason fields in the vicinity of the crack tip are obtained. The exact solutions of the stress intensity factors for the phonon field and the phason field are obtained respectively, which are very useful in practice.

2. Basic equations

When defects parallel to the quasi-periodic axis of 1D hexagonal QCs exist, the geometrical properties of the materials will be invariable along the quasi-periodic direction. In this case, the corresponding elasticity problem can be decomposed into two independent problems, i.e., a plane elasticity of conventional hexagonal crystal which can be solved by the route of the linear elastic theory [19] and an anti-plane phonon-phason field coupling elasticity problem [4]. Thus, we only need consider the latter one. The physical problem considered in this paper is shown in Fig. 1.

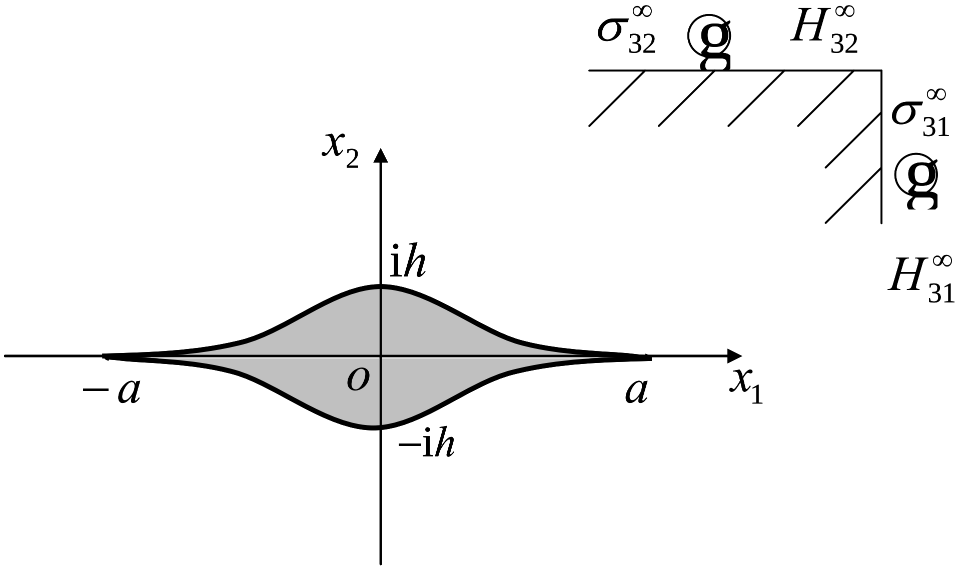

Figure 1. A lip-shape crack in 1D hexagonal QCs.

It is assumed that the quasi-periodic direction of 1D hexagonal QCs is along the positive direction of x_3 axis. In this case, all field variables are independent of x_3 and we have the following deformation geometrical equations [4]

$$\varepsilon_{3j} = \varepsilon_{j3} = u_{3,j}/2, \qquad w_{3j} = v_{3,j}, \tag{1}$$

the equilibrium equations

$$\sigma_{3j,j} = 0, \qquad H_{3j,j} = 0, \tag{2}$$

and the generalized Hooke's law

$$\sigma_{3j} = C_{44}u_{3,j} + R_3 v_{3,j}, \qquad H_{3j} = R_3 u_{3,j} + K_2 v_{3,j}, \tag{3}$$

where $j = 1, 2$; the repeated indices denote summation; a comma in the subscripts stands for a partial differentiation; σ_{ij}, ε_{3j}, u_3 are the stress, strain and displacement of the phonon field, respectively; H_{ij}, w_{3j}, v_3 are the stress, strain and displacement of the phason field; C_{44} and K_2 are the elastic constants of the phonon field and the phason field, respectively; R_3 is the phonon-phason coupling elastic constant.

Substituting Eq. (3) into Eq. (2) ,then we can obtain the following result

$$\mathbf{B}_0\nabla^2\mathbf{u} = \mathbf{0}, \tag{4}$$

where ∇^2 indicates the two-dimensional Laplace operator, and

$$\mathbf{u} = [u_3, v_3]^T,\ \mathbf{B}_0 = \begin{bmatrix} C_{44} & R_3 \\ R_3 & K_2 \end{bmatrix}, \tag{5}$$

where the superscript T represents the transpose.

Because $C_{44}K_2 - R_3^2 \neq 0$, $\mathbf{B}_0^{-1}$ exists and thus Eq. (4) is equivalent to

$$\nabla^2\mathbf{u} = \mathbf{0}. \tag{6}$$

The general solution to Eq. (6) is

$$\mathbf{u} = \mathbf{f}(z) + \overline{\mathbf{f}(z)}, \qquad z = x_1 + \mathrm{i}x_2, \tag{7}$$

where $\mathbf{f}(z)$ is an unknown complex vector; and $\overline{\mathbf{f}(z)}$ stands for the conjugate of $\mathbf{f}(z)$.

To introduction Stroh-type formalism for anti-plane deformation, we take a generalized stress function vector Σ, such that [21,22]

$$[\sigma_{31}, H_{31}]^T = -\Sigma_{,2}\quad [\sigma_{32}, H_{32}]^T = \Sigma_{,1} \tag{8}$$

Inserting Eq. (3) into Eq. (8) results in

$$-\Sigma_{,2} = \mathbf{B}_0\frac{\partial \mathbf{f}}{\partial x_1} + \mathbf{B}_0\frac{\partial \bar{\mathbf{f}}}{\partial x_1} \tag{9}$$

$$\Sigma_{,1} = \mathbf{B}_0\frac{\partial \mathbf{f}}{\partial x_2} + \mathbf{B}_0\frac{\partial \bar{\mathbf{f}}}{\partial x_2} \tag{10}$$

Eq. (9) or Eq. (10) gives

$$\Sigma = \mathrm{i}\mathbf{B}_0\mathbf{f}(z) - \mathrm{i}\mathbf{B}_0\overline{\mathrm{f}(z)} \tag{11}$$

Eqs. (7) and (11) can be rewritten as

$$\mathbf{u} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}(z)}, \tag{12}$$

$$\Sigma = \mathbf{B}\mathbf{f}(z) + \overline{\mathbf{B}\mathbf{f}(z)}, \tag{13}$$

where

$$\mathbf{A} = \mathbf{I}, \qquad \mathbf{B} = \mathrm{i}\mathbf{B_0} \tag{14}$$

where $\mathbf{I}$ is a 2×2 unit matrix.

Eqs. (12) and (13) are the general solutions of anti-plane deformations. It is seen that the stresses and stains of the phonon field and the phason field can be obtained from Eqs. (7) and (8) if the complex potential vector $\mathbf{f}(z)$ is available.

3. Stress fields and stress intensity factors

We consider a lip-shape crack in a 1D hexagonal quasicrystal solid infinitely large. It is assumed that the quasi-periodic direction of 1D hexagonal QCs is along the positive direction of x_3 axis. The solid is subjected to uniform remote anti-plane shear loadings of the phonon field and the phason field, as shown in Fig. 1 $2a$ is the crack length and $2h$ is the crack height. For the current case, we will study the complex potentials and the stress intensity factors under anti-plane shear loadings of the phonon field and the phason field at infinity. In this case, the complex function $\mathbf{f}(z)$ has the following form [21]

$$\mathbf{f}(z) = \mathbf{c}^{¥} z + \mathbf{f}_0(z), \tag{15}$$

c^{∞} is a complex constant related to the remote loading conditions, and $\mathbf{f}_0(z)$ is an unknown complex function vector, which vanishes at infinity, i.e., $\mathbf{f}_0(?\ )\quad 0$.

Differentiating Eqs. (12) and (13) with respect to x_1, we have

$$\mathbf{u}_{,1} = \mathbf{AF}(z) + \overline{\mathbf{AF}(z)}, \tag{16}$$

$$\Sigma_{,1} = \mathbf{BF}(z) + \overline{\mathbf{BF}(z)}, \tag{17}$$

where $\mathbf{F}(z) = d\mathbf{f}(z)/dz$. Substituting Eq. (15) into Eqs. (16) and (17), and then taking $z \to \infty$, results in

$$\mathbf{u}_{,1}^{\infty} = \mathbf{Ac}^{\infty} + \overline{\mathbf{A}}\overline{\mathbf{c}^{\infty}}, \tag{18}$$

$$\Sigma_{,1}^{\infty} = \mathbf{Bc}^{\infty} + \overline{\mathbf{B}}\overline{\mathbf{c}^{\infty}}, \tag{19}$$

$$\Sigma_{,1}^{\infty} = \left[\sigma_{32}^{\infty}, H_{32}^{\infty}\right]^T, \qquad \mathbf{u}_{,1}^{\infty} = \left[\varepsilon_{31}^{\infty}, w_{32}^{\infty}\right]^T. \tag{20}$$

The boundary along the surfaces of cracks is

$$\mathbf{Bf}(z) + \overline{\mathbf{Bf}(z)} = \int_s t_s ds, \qquad t_s = [t_3, h_3]^T, \tag{21}$$

where t_3 and h_3 represent the anti-plane shear traction of the phonon field and the phason field along the boundaries of cracks. In the current case, the surfaces of the cracks are free of external loadings, thus, Eq. (21) becomes

$$\mathbf{Bf}\left(z\right)+\overline{\mathbf{Bf}\left(z\right)}=0. \tag{22}$$

Inserting Eq. (15) into Eq. (22) gives

$$\mathbf{Bf}_0\left(z\right)+\overline{\mathbf{Bf}_0\left(z\right)}=-\left(\mathbf{Bc}^{\infty}z+\overline{\mathbf{B}}\overline{\mathbf{c}^{\infty}}\overline{z}\right). \tag{23}$$

In order to obtain the complex function $\mathbf{f}_0\left(z\right)$ from Eq. (23), we introduce the following mapping function [23]

$$z=\omega\left(\zeta\right)=\frac{a}{2}\rho\mathrm{i}\left[m\zeta+\frac{1}{\zeta}-\frac{\zeta}{\rho^2\left(1+m\zeta^2\right)}\right], \tag{24}$$

in which

$$\rho=\frac{1}{1-m},\qquad \beta=\frac{h}{a}=\frac{1}{2}\frac{1+m}{1-m}\left[1-\frac{\left(1-m\right)^2}{\left(1+m\right)^2}\right]^2 \tag{25}$$

$\beta<1$,the approximate representations are

$$\rho\approx1+\frac{\beta}{2}+\frac{\beta^2}{4},\qquad m\approx\frac{\beta}{2}-\frac{\beta^2}{8} \tag{26}$$

It can be shown that Eq. (24) maps the exterior region of a lip-shape crack in the z plane into the interior of a unit circle in the ζ plane, and the boundary of the lip-shape crack is transformed the unit circle τ, where we take $\omega^{-1}\left(a\right)\to-\mathrm{i}$, $\omega^{-1}\left(-a\right)\to\mathrm{i}$, as shown in Fig. 2.

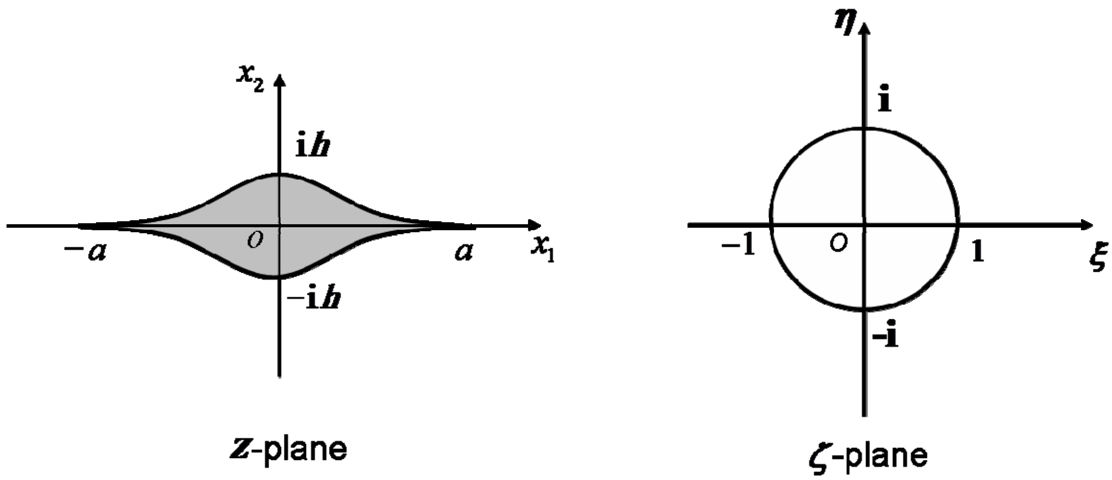

Figure 2. Mapping function of lip-shape crack into a unit circle.

Inserting Eq. (24) into Eq. (22),and then taking $\zeta = \sigma$, results in

$$\mathbf{Bf}_0(\sigma) + \overline{\mathbf{B}}\overline{\mathbf{f}_0(\sigma)} = -\left[\mathbf{Bc}^{\infty}\omega(\sigma) + \overline{\mathbf{B}}\overline{\mathbf{c}^{\infty}}\,\overline{\omega(\sigma)}\right], \tag{27}$$

where σ is the point on the unit circle, and $\mathbf{f}_0(\sigma) = \mathbf{f}_0(\omega(\sigma))$ is defined.

Multiplying Eq. (27) by $d\sigma/[2\pi\mathrm{i}(\sigma-\zeta)]$, where ζ is an arbitrary point inside the unit circle, and performing the Cauchy integral along the unit circle τ in the anticlockwise direction, we have

$$\mathbf{Bf}_0(\zeta) = -\mathbf{Bc}^{\infty}\frac{1}{2\pi\mathrm{i}}\int\frac{\omega(\sigma)}{\sigma-\zeta}d\sigma - \overline{\mathbf{B}}\overline{\mathbf{c}^{\infty}}\frac{1}{2\pi\mathrm{i}}\int_{\tau}\frac{\overline{\omega(\sigma)}}{\sigma-\zeta}d\sigma \tag{28}$$

$\omega(\zeta)$ is analytic inside the unit circle, except for a simple pole at $\zeta = 0$. Since $\overline{\omega(\zeta)}$ is analytic inside the unit circle, except for the simple poles at $\zeta = 0$, $\zeta = \sqrt{m}\mathrm{i}$, $\zeta = -\sqrt{m}\mathrm{i}$. By the residue theorem in complex variable function, one has

$$\frac{1}{2\pi\mathrm{i}}\int\frac{\omega(\sigma)}{\sigma-\zeta}d\sigma = \omega(\zeta) - \frac{a}{2}\rho\mathrm{i}\frac{1}{\zeta}, \tag{29}$$

$$\frac{1}{2\pi\mathrm{i}}\int_{\tau}\frac{\overline{\omega(\sigma)}}{\sigma-\zeta}d\sigma = \overline{\omega(\zeta)} + \frac{a}{2}m\rho\mathrm{i}\frac{1}{\zeta} - \frac{a}{2}\mathrm{i}\frac{\zeta}{\rho\left(m+\zeta^2\right)}. \tag{30}$$

The following result can be obtained by Eq.(18),Eq.(19),Eq.(20)

$$\mathbf{Bc}^{\infty} = \frac{1}{2}\left\{\left[\sigma_{32}^{\infty}, H_{32}^{\infty}\right]^T + \mathrm{i}\left[\sigma_{31}^{\infty}, H_{31}^{\infty}\right]^T\right\}, \tag{31}$$

$$\overline{\mathbf{B}}\overline{\mathbf{c}^{\infty}} = \frac{1}{2}\left\{\left[\sigma_{32}^{\infty}, H_{32}^{\infty}\right]^T - \mathrm{i}\left[\sigma_{31}^{\infty}, H_{31}^{\infty}\right]^T\right\}, \tag{32}$$

Substituting Eqs. (31) and (32) into Eq. (28), then differentiating the obtained results with respect to ζ leads to

$$\mathbf{BF}_0(\zeta) = \frac{a\rho}{4}\left[m+1-\frac{1-m\zeta^2}{\rho^2\left(1+m\zeta^2\right)^2}\right]\begin{bmatrix}\sigma_{31}^{\infty}\\ H_{31}^{\infty}\end{bmatrix} - \frac{a\rho}{4}\left[m-1-\frac{1-m\zeta^2}{\rho^2\left(1+m\zeta^2\right)^2}\right]\begin{bmatrix}\sigma_{32}^{\infty}\\ H_{32}^{\infty}\end{bmatrix} \tag{33}$$

$\mathbf{F}_0(\zeta) = d\mathbf{f}_0(\zeta)/d\zeta$.

$\omega'(\zeta)$ and $\left[\overline{\omega(\zeta)}\right]'$ can be given by Eq. (24) as follows

$$\omega'(\zeta) = \frac{a}{2}\rho\mathrm{i}\left[m-\frac{1}{\zeta^2}-\frac{1-m\zeta^2}{\rho^2\left(1+m\zeta^2\right)^2}\right], \tag{34}$$

$$\left[\overline{\omega(\zeta)}\right]' = -\frac{a}{2}\rho\mathrm{i}\left[1-\frac{m}{\zeta^2}-\frac{m-\zeta^2}{\rho^2\left(m+\zeta^2\right)^2}\right], \tag{35}$$

It is found from Eq. (33) that the stress fields can be obtained from the relations between the stress and stress function.

The stress intensity factor at the crack tip is a very important physical quantity in fracture mechanics, which can reflect the stress intensity around the crack tip. The vector of the stress intensity factors can be defined as

$$\boldsymbol{k} = \left[K_{III}^h, K_{III}^{\perp}\right]^T = \lim_{z\to z_1}\sqrt{2\pi\left(z-z_1\right)}\,\Sigma_{,1}, \tag{36}$$

where K_{III}^h and $K_{III}^{\perp}$ denote the stress intensity factors of the phonon field and the phason field, respectively.

Substituting Eq. (17) into Eq. (36) gives

$$\boldsymbol{k} = \left[K_{III}^h, K_{III}^{\perp}\right]^T = 2\lim_{z\to z_1}\sqrt{2\pi\left(z-z_1\right)}\mathbf{BF}_0\left(z\right) \tag{37}$$

in which the condition that $\mathbf{BF}_0\left(z\right)$ is imaginary along the x_2 axis is used.

In the ζ plane, Eq. (37) becomes

$$\boldsymbol{k} = 2\sqrt{2\pi}\lim_{\zeta\to-\mathrm{i}}\sqrt{\omega(\zeta)-\omega(-\mathrm{i})}\,\frac{\mathbf{BF}_0\left(\zeta\right)}{\omega'\left(\zeta\right)}, \tag{38}$$

where $\zeta = -\mathrm{i}$ is the corresponding point of the crack tip $z = a$.

It is obvious from Eqs. (33) and (34) that one finds $\lim_{\zeta\to-\mathrm{i}}\omega'\left(\zeta\right)$ exists and $\lim_{\zeta\to-\mathrm{i}}\mathbf{BF}_0\left(\zeta\right)\neq 0$. Thus, by the L'Hospital rule, Eq.(38) results in

$$\boldsymbol{k} = 2\sqrt{2\pi}\lim_{\zeta\to-\mathrm{i}}\frac{\mathbf{BF}_0\left(\zeta\right)}{\sqrt{\omega''\left(\zeta\right)}}, \tag{39}$$

$\omega''(\zeta)$ can be obtained by Eq.(34)

$$\omega''(\zeta) = \frac{a}{2}\rho\mathrm{i}\left[\frac{2}{\zeta^3}-\frac{2m^2\zeta^3-6m\zeta}{\rho^2\left(1+m\zeta^2\right)^3}\right], \tag{40}$$

Inserting Eqs. (33) and (40) into Eq. (39), the analytic expressions of the stress intensity factors at the crack tip $\left(a,0,0\right)$ for the anti-plane shear problem are derived as follows

$$\boldsymbol{k} = \begin{bmatrix}\sigma_{32}^{\infty}\\ H_{32}^{\infty}\end{bmatrix}\sqrt{\pi a}K, \tag{41}$$

where

$$K = \sqrt{\frac{\rho}{1+3m\rho+m^2\rho}}. \tag{42}$$

If the crack height h tends to zero, one has $m=0,\ \rho=1$ and then Eq. (41) reduces to

$$K = \sqrt{\frac{\rho}{1+3m\rho+m^2\rho}} = 1. \tag{43}$$

which is the solution of Griffith cracks in a 1D hexagonal QC [15].

4. Numerical examples

We consider the variation of K with $\beta = \frac{h}{a}$. It can be shown from Fig. 3 that if $\beta < 1$, the dimensionless field intensity factor K decreases with the value of β becomes large. It indicates that an increase of the height of the tip-shape crack will retard the crack propagation. In particular, when the height of the lip-shape crack approaches to zero, the current case can be reduced to the Griffith crack, i.e., $K=1$, which is easier to propagate.

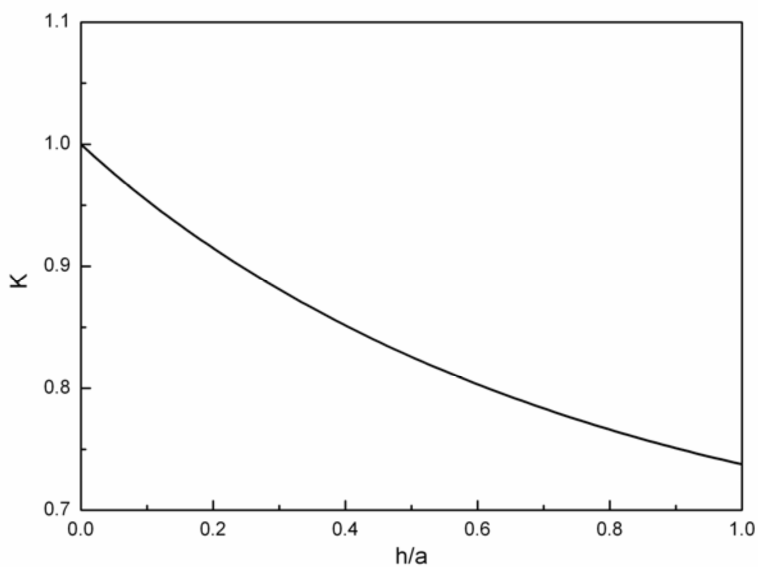

Figure 3. Variation of K with $\beta = \frac{h}{a}$

Acknowledgements

The authors thank the support from the National Natural Science Foundation of China (Nos.11262012 and 11262017), the Scientific Research Key Program of Inner Mongolia University of Technology (Grant No. ZD201219).

References

[1] D. Shechtman, I. Blech, D. Gratias, J.W. Cahn, Metallic phase with long-range orientational

order and no translational symmetry, Phys. Rev. Lett 53(1984) 1951－1953.
[2] S.B. Rochal, Second-order terms of the phonon－phason dynamic matrix of an icosahedral quasicrystal: Diffuse intensity and the profile shape around the Bragg peaks, Phys. Rev. B 64 (2001) 144204.
[3] S.B. Rochal, V.L. Lorman, Minimal method of the phonon－phason dynamics in icosahedral quasicrystals and its application to the problem of internal friction in the i-AIPdMn alloy, Phys. Rev. B 66 (2002) 144204.
[4] T.Y. Fan, The Mathematical Theory of Elasticity of Quasicrystals and its Applications, Science Press, Beijing/Springer-Verlag, Heidelberg, 2010.
[5] P De and R A Pelcovits, Phys. Rev. B35, 8609 (1987)
[6] W G Yang, R H Wang, D H Ding and C Z Hu, Phys. Rev. B48, 6999 (1993)
[7] Ding, D.H., Yang, W.G., Hu, C.Z., Wang, R.H., 1993. Generalized elasticity theory of quasicrystals. Phys. Rev. B 48, 7003－7010.
[8] R H Wang, W G Yang, C Z Hu and D H Ding, J. Phys. Condens. Matter 9, 2411(1997)
[9] Liu, G.T., Fan, T.Y., Guo, R.P., 2004. Governing equations and general solutions of plane elasticity of one-dimensional quasicrystals. Int. J. Solids Struct. 41,3949－3959.
[10] X.F. Li, T.Y. Fan New method for solving elasticity problems of some planar quasicrystals and Chin. Phys. Lett., 15 (1998), pp. 278–280
[11] Y.Z. Peng, T.Y. Fan Perturbation method solving elastic problems of icosahedral quasicrystals containing circular crack Chin. Phys., 9 (10) (2000), pp. 764–766
[12] X.M. Meng, B.Y. Tong, Y.K. Wu, Mechanical properties of Al65 Cu20 Co15, Acta Mech. Sin. 30 (1994) 60－64 (in Chinese).
[13] Z. Zhang, K. Urban, Transmission electron microscope observation of dislocation and stackling faults in a decagonal Al-Cu-Co alloy, Phil. Mag. Lett. 60 (1989) 97－102.
[14] T.Y. Fan, X.F. Li, Y.F. Sun, A moving screw dislocation in a one-dimensional hexagonal quasicrystals, Phys. Sin. Overseas Edition 8 (1999) 288－295.
[15] T.Y. Fan, 1999 The Mathematical Theory of Elasticity of Quasi-crystals and Applications, Beijing:Beijing Institute of Technology Press. (in chinese)
[16] L.H. Li, T.Y. Fan, Exact solutions of two semi-infinite collinear cracks in a strip of one dimensional hexagonal quasicrystal, Appl. Math. Comput. 196(2008) 1－5.
[17] J.H. Guo, G.T. Liu, Analytic solutions of the one-dimensional hexagonal quasicrystals about problem of a circular hole with asymmetry cracks, J. Appl. Math. 30 (2007) 1066－1075 (in Chinese).
[18] J.H. Guo, G.T. Liu, Analytic solutions to problem of elliptic hole with two straight cracks in one-dimensional hexagonal quasicrystals, Appl. Math. Mech. 29 (2008) 485－493.
[19] J.H. Guo, G.T. Liu, Exact analytic solutions for an elliptic hole with asymmetric collinear cracks in a one-dimensional hexagonal quasi-crystal, Chin. Phys. B 17 (2008) 2610－2620.
[20] N.I. Muskhelishvili, Some Basic Problems of Mathematical Theory of Elasticity, Noordhoff, Groningen, 1963.
[21] T.Y. Zhang, C.F. Gao, Fracture behaviors of permeable multi-cracks in a piezoelectric material, Theor. Appl. Frac. Mech. 41 (2004) 339－379.
[22] J.H. Guo, Z.X. Liu, Exact solution of four cracks originating from an elliptical hole in one-dimensional hexagonal quasicrystals. Appl. Math. Comput. 217 (2011) 9397-9403

[23] Z.B. Kuang, Stress analysis for plane curved polygonal defects containing cusps only, Chin. J. Theo. Appl. Mech., 14 (1979)218-228. (in chinese)