Instability and failure of particulate materials caused by rolling of non-spherical particles

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Abstract In particulate materials under compression at the peak load the accumulated damage allows particle rolling. For non-spherical particles the moment equilibrium dictates that further increase in displacement requires reduced shear stress producing an effect of apparent negative stiffness; its value depends upon the magnitude of the compressive stress. Dilatancy produced by rolling particles reduces the value negative stiffness, while the contraction phase causes immediate instability. Material with rolling particles is macroscopically modelled as a matrix containing inclusions with negative shear modulus. When the concentration of negative stiffness inclusions is low, the effective shear modulus is positive and the material is stable. When the concentration reaches a critical level the effective shear modulus abruptly becomes negative and the material loses stability. Furthermore, there exists a special value of negative shear modulus of inclusions (and hence the magnitude of compressive stress) when the critical concentration becomes zero, such that the first rolling particle induces the global instability.

Keywords Rolling particles, Negative shear modulus, Effective shear modulus, Critical concentration, Dilation

1. Introduction

The importance of particle rotations (and the associated rotational degrees of freedom) in the mechanisms of instability and failure of particulate materials has long been recognised (e.g. [1-7]). The particle rotations were observed in physical experiments (e.g. [8-10]) and discrete element simulations (e.g. [11-14]). The modelling of the effect of particle rotation was mainly based on the concept of spherical (circular in 2D) particles, which offered the maximum simplicity. The effect of particle shape was thought to be quantitative, for instance resulting in reduced velocities of particle flow and increased stresses (e.g. [15]). However, the non-spherical particles can interlock – a phenomenon that does not exist in spherical particles [16]. Furthermore, rotations of non-spherical particles cause elbowing [17] that is coupling between the rotations and normal stresses. Both these mechanisms could lead to qualitatively new phenomena, such as the apparent negative stiffness [18-23]. It was further pointed out in [24] that in producing the negative stiffness effect the role of non-spherical particles could be played by clusters of connected spherical particles. The role of non-spherical particles and clusters of spherical particles is also discussed in [25].

Dyskin and Pasternak [20-24] modelled the apparent negative stiffness associated with particle rotations without taking into account the effect elbowing has on dilation/contraction. Here we include the latter into consideration. This will be accomplished in Section 2. Section 3 models the volume elements with apparent negative stiffness as inclusions in a matrix with positive definite elastic moduli and takes into account the interaction between the inclusions. This result gives an insight into the effect of rotating particles on global stability.

2. Apparent negative stiffness caused by rotating particles. The effect of elbowing

Consider a particulate material loaded in compression to near the peak load. We assume that in the process of loading considerable amount of defects have been accumulated, mainly on the bonds (cement) between the particles such that some particles are now partially detached from the matrix, Fig. 1. The balance of moments of shear and normal forces shown in Fig. 1 about point O reads

$$T\sin\varphi + P\cos\varphi = 0, \quad \pi/2 \le \varphi \le \pi\,. \tag{1}$$

Here l is the corresponding particle diameter, T and P are the magnitudes of the shear and normal forces and angle φ is related to the position of the particle at the moment of detachment. Obviously, the moment equilibrium is only possible for the range of angles φ indicated in (1) and in Fig. 1a; when $\varphi<\pi/2$, Fig. 1b, the particle becomes unstable. It is reasonable to assume that the initial packing of particles was a stable one similar to the configuration shown in Fig. 1a. The analysis below is based in infinitesimal deformations and hence all movements considered will leave the initially stable configuration in its stable state. For that reason we will disregard the unstable configurations shown in Fig. 1b.

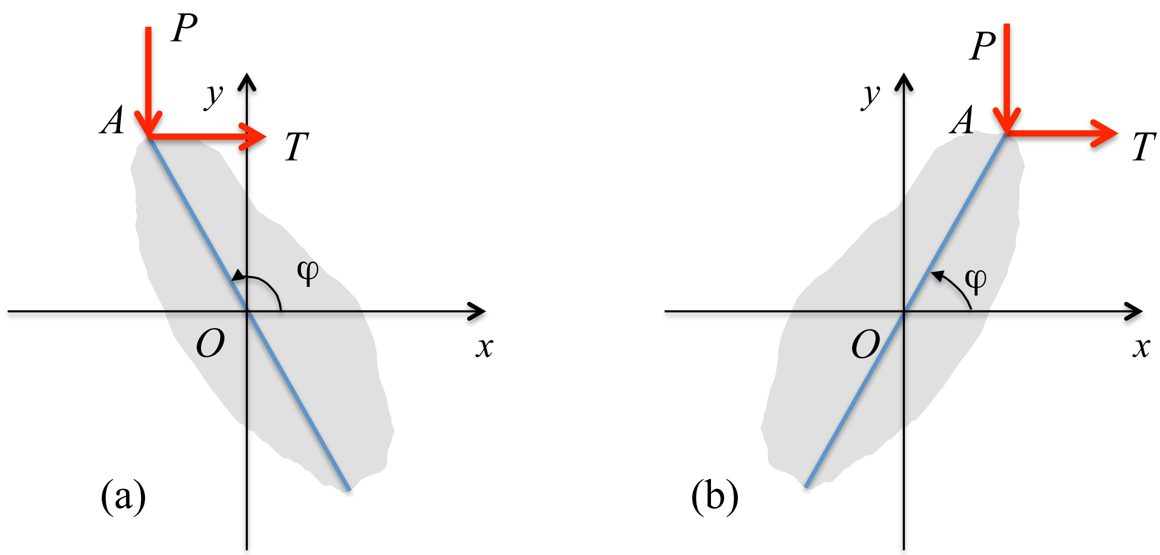

Figure 1. A moment balance of rotating (rolling) partially detached particle: (a) stable configuration, (b) unstable configuration.

Suppose the particle underwent an infinitesimal rotation $d\varphi$. This will change the coordinates (x, y) of point A, Fig. 1 by dx=-$l\sin\varphi\, d\varphi$ and dy=$l\cos\varphi\, d\varphi$. The moment equilibrium (1) imposes the following force increments:

$$dT\sin\varphi - T\frac{\cos\varphi}{\sin\varphi}\frac{dx}{l} + dP\cos\varphi - T\frac{\sin\varphi}{\cos\varphi}\frac{dy}{l} = 0\,. \tag{2}$$

We now assume that the change dP in the magnitude of compression is proportional to the vertical displacement such that

$$dP = k_m\frac{dy}{l}\,. \tag{3}$$

where k_m is the stiffness of the surrounding parts of the particulate material. Hereafter we refer to the surrounding parts of the particular material as the *matrix*.

Substituting (3) into (2), expressing T via P through the equation of moment equilibrium (1) and taking into account that dy=-dx $\cos\varphi/\sin\varphi$ we obtain

$$dT = \left\{-P\frac{1}{\sin^3\varphi} + k_m\frac{\cos^2\varphi}{\sin^2\varphi}\right\}\frac{dx}{l}\,. \tag{4}$$

It is seen that the coefficient between incremental shear force dT and incremental shear strain dx/l can assume negative values, when $P>k_m\sin\varphi\cos^2\varphi$. We call this effect the *apparent negative stiffness*.

Following [20-23] we model the collective effect of rotating particles by treating them as negative stiffness inclusions (inclusions with negative shear modulus, μ_{incl}) embedded in a matrix with positive definite elastic moduli. We then use the theory of effective characteristics in order to determine the elastic moduli of such a composite at macroscale and determine the conditions of global instability. In order to incorporate this phenomenon into a continuum description of the granulate material consider a representative volume element, that is an element of size $H>>l$. We introduce normal p and shear τ stresses acting on the faces of the element. Therefore we can treat normal and shear forces from (4) as $P\sim pl^2$ and $T\sim\tau l^2$. For the sake of simplicity we will treat the normalised matrix stiffness as the bulk modulus of the matrix, $\kappa_m\sim k_m/l^2$. Then we can express the average shear modulus, μ_{incl}, associated with the particle rotation as

$$\mu_{incl} = -\frac{p}{\sin^3\varphi} + \kappa_m\frac{\cos^2\varphi}{\sin^2\varphi}\,. \tag{5}$$

Here angle φ is interpreted as an average angle, which provides a combined description of particle shapes and initial packing.

The negative stiffness is achieved when

$$p > p_{\min}(\varphi) = \kappa_m \sin\varphi \cos^2\varphi \,. \tag{6}$$

Dependence (6) and its interpretation are shown in Fig. 2. It is seen that the dense packing produces negative stiffness at lower pressures that the loose one.

Two interim conclusions could be made here. Firstly, the compressive stress required to produce the effect of apparent negative stiffness could be of the order of the bulk modulus of the surrounding rock. That is only possible when the bulk modulus is sufficiently reduced by the damage accumulated in the preceding loading, namely near the peak load. Another conclusion is that the phenomenon of apparent negative stiffness depends upon the density of the initial packing.

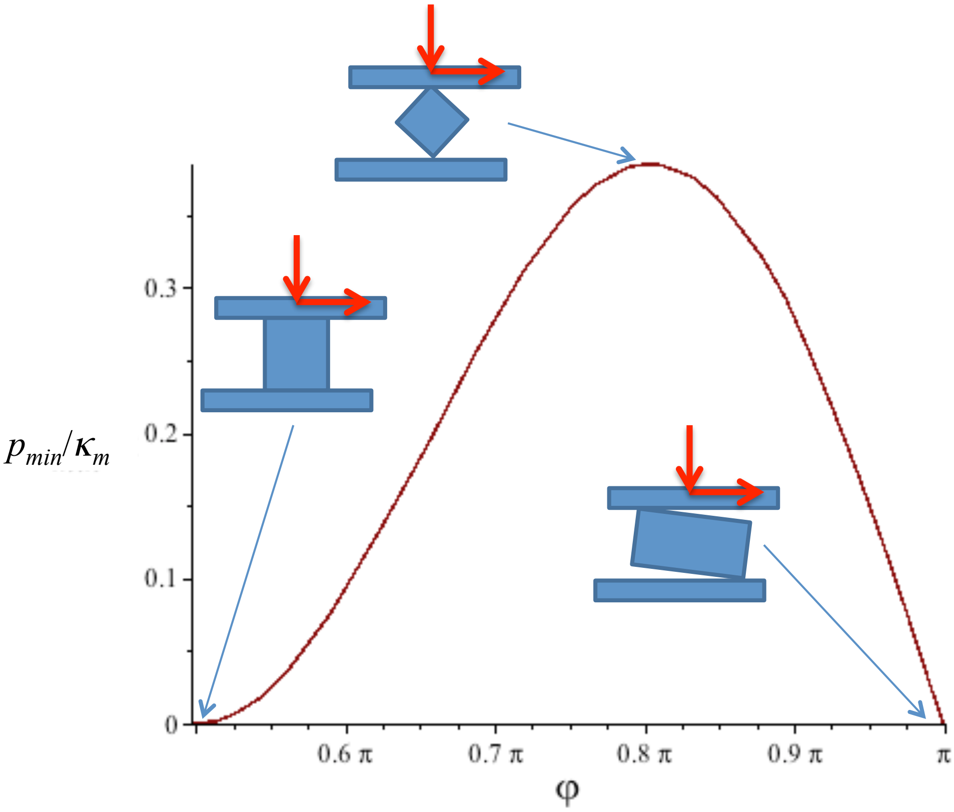

Figure 2. Dependence (6) and its interpretation: the smallest values of the minimal compressive stress magnitude p_{min} needed to ensure the negative stiffness are achieved at dense packing (angles close to $\pi/2$ and π), while the loose packing (angles close to $3\pi/4$) requires higher compressive stresses to show negative stiffness.

3. Effect of rotating particles on global stability

The presence of negative stiffness means the loss of positive definiteness of the tensor of elastic moduli (or the quadratic form of elastic energy) and hence the loss of uniqueness of the elastic solution. The material with negative stiffness becomes intrinsically unstable; its actual stability or instability depends upon the boundary conditions, i.e. the type of loading applied. The best known example of this type of dependence upon the boundary conditions is the post-peak loading of rock or concrete sample which is only possible if the loading device is sufficiently stiff and the loading is displacement controlled.

In the case under consideration the negative stiffness associated with particle rotation/rolling is a local phenomenon. We model it as a negative stiffness (negative shear modulus) inclusion in a conventional matrix. The stability/instability of the inclusion should depend upon the deformability of the matrix. This suggests that the global stability of the material with rotating particles could be determined by considering the matrix filled with negative stiffness inclusions and determining the effective moduli of such a composite. If the tensor of effective moduli is positive definite, the particulate material is stable, otherwise it is unstable.

The theory of effective characteristics of a matrix with negative stiffness inclusions was developed in [23, 26]. Following [23] we model the negative stiffness inclusions as spherical inclusions with negative shear modulus given by (5) and the (positive) bulk modulus, κ_{incl}, equal to that of the matrix, $\kappa_{incl} = \kappa_m$. The shear modulus of the matrix, μ_m is assumed to be positive.

The simplest case usually considered in the theory of effective characteristics is the case of low concentrations (volumetric fractions) of inclusions. In this case the interaction between the inclusions can be neglected and the problem of determination of the effective moduli is reduced to the determination of the change in the average strain (volumetric averaging over representative volume element is presumed) caused by a single inclusion under uniform stress. This is so-called approximation of low concentrations. It was shown in [27] and then in [23, 26] that the application of the approximation of low concentrations to the matrix with negative stiffness inclusions leads to a singularity, which in our case happens at a certain critical value of the negative shear modulus. The singularity means that the influence of the inclusions on the effective moduli is infinite at any concentrations. The existence of the singularity can be interpreted as the increasing influence of interaction between the inclusions as the shear modulus of inclusion tends to that (negative) value. Therefore one has to use the methods of computing effective moduli that account for the interaction.

We use the differential self-consistent method, which in the case of isotropic matrix with spherical inclusions of any concentrations c leads to the following system of differential equations obtained in [28]:

$$\begin{cases} \dfrac{d\kappa_{eff}}{dc} = \dfrac{\kappa_{incl} - \kappa_{eff}}{1-c} \dfrac{\kappa_{eff} + \kappa^*}{\kappa_{incl} + \kappa^*}, \quad \kappa^* = \dfrac{4}{3}\mu_{eff} \\ \dfrac{d\mu_{eff}}{dc} = \dfrac{\mu_{incl} - \mu_{eff}}{1-c} \dfrac{\mu_{eff} + \mu^*}{\mu_{incl} + \mu^*}, \quad \mu^* = \dfrac{\mu_{eff}}{6} \dfrac{9\kappa_{eff} + 8\mu_{eff}}{\kappa_{eff} + 2\mu_{eff}} \\ \kappa_{eff}\big|_{c=0} = \kappa_m, \quad \mu_{eff}\big|_{c=0} = \mu_m \end{cases} . \tag{7}$$

where c is the volumetric fraction (concentration) of inclusions.

In our case, $\kappa_{incl} = \kappa_m$ and therefore $\kappa_{eff} = \kappa_m$ is an obvious solution of the first equation in (7). Due to the uniqueness of the solution of this system of differential equations, there are no other solutions.

We now normalise the moduli with κ_m by formally assuming that κ_m =1. After introducing the notations

$$\mu_{eff} = \mu, \quad \mu_{incl} = -m\mu_m, \quad m = \frac{p}{\sin^3\varphi} - \frac{\cos^2\varphi}{\sin^2\varphi}. \tag{8}$$

system (7) is reduced to

$$\begin{cases} \dfrac{d\mu}{dc} = \dfrac{-5\mu}{1-c} \cdot \dfrac{(4\mu+3)(m\mu_m + \mu)}{8\mu^2 - 3(4m\mu_m - 3)\mu - 6m\mu_m} \\ \mu\big|_{c=0} = \mu_m \end{cases} . \tag{9}$$

Derivative $d\mu/dc$ is discontinuous when the denominator in the right hand site of (9) vanishes. The discontinuities correspond to points μ_1 and μ_2:

$$\mu_{1,2} = \frac{\sqrt{3}}{16}\left[(4m\mu_m - 3)\sqrt{3} \pm \sqrt{48\mu_m^2 m^2 - 8m\mu_m + 27}\right]. \tag{10}$$

It can be shown that $\mu_2<0<\mu_1$. Since the initial condition in (9) is $\mu(0)=\mu_m>0$, the solution of (9) can only reach point μ_1, after which the effective shear modulus drops to a certain negative value determined by the global loading device which applies the load to the particulate material [23, 26]. We therefore treat the modulus μ_1 as a point of global intrinsic instability of the particulate material.

Solution of (9) can be obtained in the following implicit form

$$\frac{\left(\mu+m\mu_m\right)^5}{\mu^2\left(4\mu+3\right)}=\mu_m^3\frac{\left(1+m\right)^5}{4\mu_m+3}\left(1-c\right)^5. \tag{11}$$

The point of instability is reached when $\mu=\mu_1$. This happens at the concentration of negative stiffness inclusions

$$c_{cr}=1-\frac{\mu_1+m\mu_m}{1+m}\left[\frac{4\mu_m+3}{\mu_m^3\mu_1^2\left(4\mu_1+3\right)}\right]^{1/5}. \tag{12}$$

The dependencies (11) and (12) are shown in Figs. 3 and 4 respectively for different values of m and μ_m. It is seen from Fig. 3 that the effective shear modulus can both increase and decrease with concentration of the negative stiffness inclusions depending upon the values of parameters m and μ_m. The plot of critical concentration, Fig. 4a, shows that there exist combinations of parameters m and μ_m at which the critical concentration is zero. That means that at the instance when the particle rotations start and make the corresponding shear modulus negative, the particulate material loses stability. Dependence of the value of negative shear modulus of inclusions vs. the shear modulus of the matrix is shown in Fig 4b. It is seen that the dependence is relatively weak; the value of the negative shear modulus that delivers zero critical concentration is of the order of the shear modulus of the matrix.

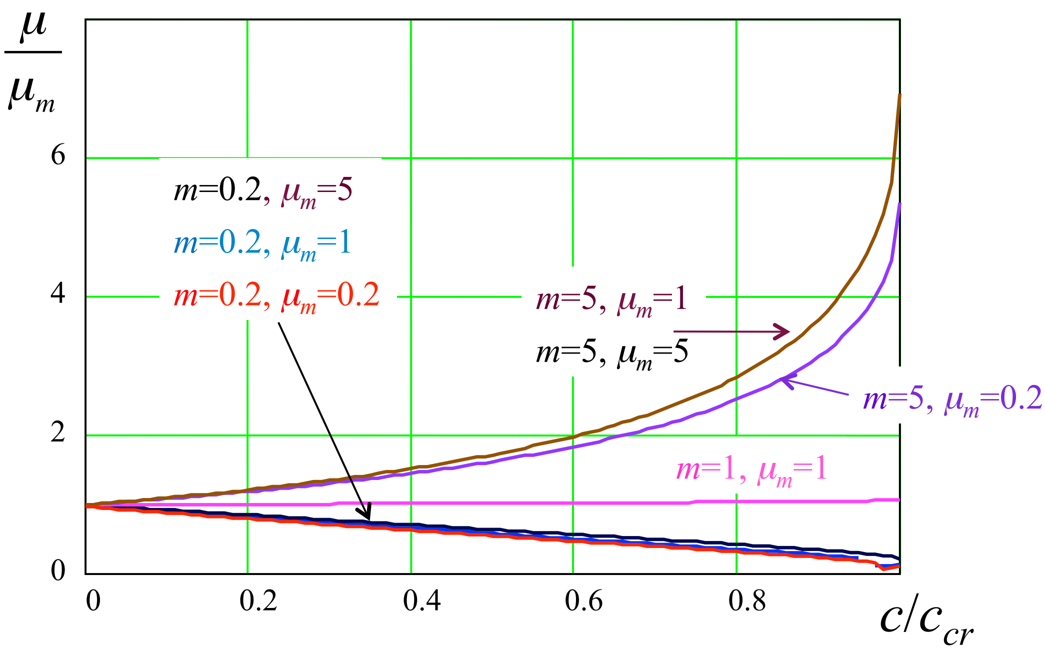

Figure 3. Effective shear modulus μ vs. volumetric fraction (concentration) of negative stiffness inclusions c. Three pairs of parameters m and μ_m on the left side of the picture refer to nearly indistinguishable dependencies in the same order from top to bottom.

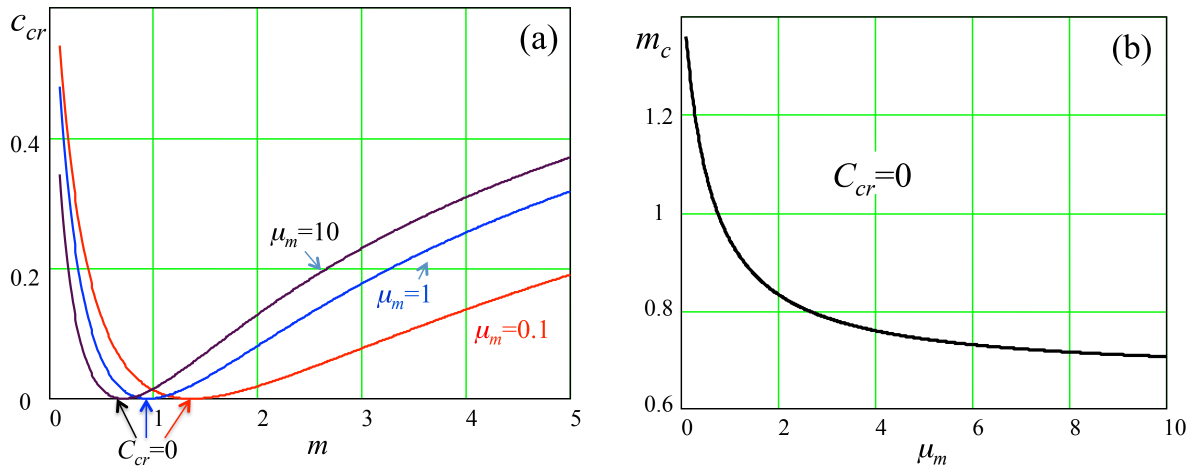

Figure 4. The critical concentration of negative stiffness inclusions c_{cr}: (a) its dependence upon the value m of negative shear modulus of inclusions; (b) the values m_{cr} of negative shear modulus of inclusions delivering zero critical concentration.

From here it is seen that the magnitude of compressive stress that produces the global instability of the particulate material is

$$p_{cr}/\kappa_m = m_c \sin^3\varphi + \sin\varphi\cos^2\varphi\ . \tag{13}$$

4. Conclusions and outlook

The ability of partially detached non-spherical particles to roll or rotate leads to the effect of apparent negative stiffness (negative shear modulus), whose value depends on the magnitude of the applied compressive stress. This is a property of particle non-sphericity: rotation of spherical (or circular in 2D) particles does not produce negative stiffness. Rotation of non-spherical particles also produces elbowing which results in dilation of the surrounding material. Depending of the initial packing, dilation can lead to the reduction of the value of negative shear modulus such that the magnitude of compressive stress needed to effect negative stiffness is of the order of the bulk modulus. Therefore the effect of negative stiffness is only relevant to the particulate materials loaded in compression up to the peak when the damage created in the course of loading has considerably weakened the material and made the moduli sufficiently low.

The global instability of the particulate material with rolling or rotating particles is reached when the effective shear modulus is no longer positive. This happens when the concentration (volumetric fraction) of negative stiffness areas reaches a certain critical value that depends upon the value of negative shear modulus and the shear modulus of the surrounding material. There exist a combination of these parameters which makes the critical concentration zero, meaning that the first rolling particle results in global instability.

The theory proposed casts light on the mechanics of compressive failure of particular materials such as rock and concrete as well as on the mechanism instability of granular materials. Another possible application of this theory is in the design of a special class of hybrid materials based on specially shaped particles or blocks to ensure the desirable properties of the hybrid not achievable otherwise. In particular, according to Fig. 3 the presence of rotating non-spherical particles in a matrix can either increase or decrease the effective shear modulus depending upon the magnitude of applied compressive stress. This suggests a method of designing materials whose moduli can be controlled by applied load without the creation of additional internal damage.

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