Random elasto-plastic lattice modeling of damage in fibrous materials

Keqiang Hu[1,*], Zengtao Chen[1], K. C. Li[2], X. Frank Xu[3]

[1] Department of Mechanical Engineering, University of New Brunswick, Fredericton, NB, Canada E3B 5A3
[2] Department of Chemical Engineering, University of New Brunswick, Fredericton, NB, Canada E3B 5A3
[3] School of Civil Engineering, Beijing Jiaotong University, Beijing 100044, China
* Corresponding author: ckhu@unb.ca, keqianghu@163.com

Abstract A random elasto-plastic lattice network model is developed to simulate the damage behavior of fibrous materials. Elasto-plastic bar elements are used to construct a regular lattice network with the random element strength distributions to simulate the randomness of the continuum fibrous materials. The structural response of the elasto-plastic lattice network under displacement controlled loading is studied using finite element method. Small deformation theory is used and the modified Newton-Raphson algorithm is applied to solve the nonlinear finite element equations. The effects of correlation length of the strength distribution of bar elements on the global behaviors are studied.

Keywords Damage, Random elasto-plastic lattice, Nonlinear finite element method, Correlation length

1. Introduction

Complexity of failure is reflected from sensitivity of strength to small defects and wider scatter of macroscopic behaviors. The information of materials at micro-scale is random and can only be partially measurable, which leads to the complicated failure mechanisms for the random heterogeneous materials [1-3]. Various types of lattice type models such as central force model, electrical fuse model, bond-bending model, and beam-type model have been used to study progressive damage of heterogeneous materials, such as concrete, rock, ceramics and paper. It is a relatively simple but powerful technique to identify microcracking, crack branching, crack tortuosity and bridging, thus allowing the fracture process to be followed until complete failure [4-8]. A comprehensive review of the lattice models for micromechanics applications can be found in Ostoja-Starzewski [9].

The failure properties of fibrous materials have been a subject of research for the past decades [10-12]. As the structure of fibrous material is inhomogeneous, the role of disorder has great influence on the mechanical and rheological properties, and statistical growth models can simulate the heterogeneous fibrous materials well. The fibers in fibrous materials are full of imperfections and exhibit a wide variety, of natural origin, in dimensions and mechanical properties, and the mechanical properties of fibrous materials can be varied significantly by selecting different types of fibers [13]. Paper, a material known to everybody, has a fibrous network structure consisting of wood fibers. A random geometry fiber network model [14] was considered to study the special elastic orthotropy of machine-made papers, which has anisotropy in the two principal directions, the machine direction (MD) and the cross direction (CD). It was shown that the random geometry may lead to a macroscopic property of special elastic orthotropy [14]. A two-dimensional beam network model was proposed as a micromechanics model to simulate paper's failure process due to sequential breakages of fibers and/or bonds, and the numerical results showed the effects of fiber length and the ratio of fiber strength to bond strength on the failure characteristics of paper [7].

In this paper, a random elasto-plastic lattice model is proposed according to the equivalence of strain energy instead of the true network structure in fibrous materials. The concept of unit cell is adopted to calibrate the material properties of the network based on those of the continuum paper. To characterize the heterogeneity of the microstructure of fibrous materials, the yield strength of the elasto-plastic fiber element is considered to follow a correlated random distribution. Nonlinear finite element method is utilized to study the structure response of the lattice under external tension loading. The lattice network can be considered as a stochastic representative volume element (SRVE), which transports the local effects into global solutions with uncertainty information, e.g. probability distribution. The effect of the correlation length on the strength of the SRVE is studied.

2. Random lattice model

To simulate the damage of fibrous materials, one of the most effective numerical approaches is to use the lattice model that allows disorder to be introduced naturally. Since fibrous material microstructure is extremely complicated, as shown in Figure 1, it is very hard to construct a numerical network exactly the same as the true physical one. Therefore, the idea of adopting a regular lattice equivalent to macro-level continuum in terms of strain energy [8, 9, 15, 16], is applied to study the damage of fibrous materials. For simplicity, the regular triangular network, as shown in Figure 2, is used in this work to study the failure properties of fibrous material structures. The fibers are distributed in three different directions with an increment of $\pi/3$, which leads to the isotropic properties of the structure when all fibers are assigned the uniform properties [8, 9]. In the framework of finite element method, the nodes correspond to fiber-to-fiber bonds, while two-node elements are formed by fiber segments between every two neighboring nodes.

To simulate the randomness of the microstructure of fibrous materials, we need to generate random field (RF) samples according to given probability distributions. In this study, non-Gaussian RF samples are generated from underlying Gaussian RF samples by the so-called translation method. An overview of the random field simulation is presented in [17]. For simplicity, bar-elements are used to construct the regular lattice network, and there are two translation degrees of freedom for each node. The bar elements are considered elasto-plastic and their yield strength is assumed to follow a correlated random Weibull distribution [1]. A Weibull RF sample Y can be generated from an underlying Gaussian RF sample X via [1]

$$Y = F_W^{-1}(F_g(X)) \tag{1}$$

where $F_g(\cdot)$ is the standard normal cumulative density function (cdf), and $F_W^{-1}(\cdot)$ is the inverse of the Weibull cdf. The correlation function of the underlying Gaussian RF is assumed to be

$$\rho(x, y) = \exp\left[-\left(x^2 + y^2\right)/d^2\right] \tag{2}$$

where d indicates the correlation length.

As a rule of thumb, an average tensile strength of fibrous materials can be chosen as equal to elastic modulus times $(1.0 \pm 0.1)\%$ [13]. The generated random Weibull distribution with a set mean value is mapped to the regular lattice network to characterize the heterogeneity of the microstructure. For example, two of the Weibull RF samples are shown in Figure 3 for $d = 1$ and $d = 4$, respectively. In Figure 3, the strength values are normalized by the mean value of the samples, and $d = 1$ denotes that the correlation length is equal to the element length. It is shown that as the correlation length d increases, the points over the range d are more likely correlated.

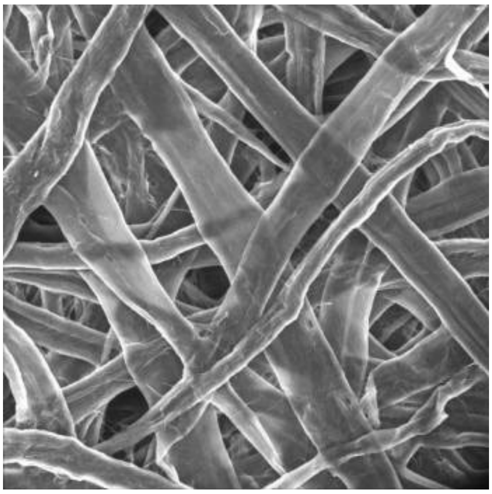

Figure 1. Scanning electron microscope image of a fibrous material structure [12].

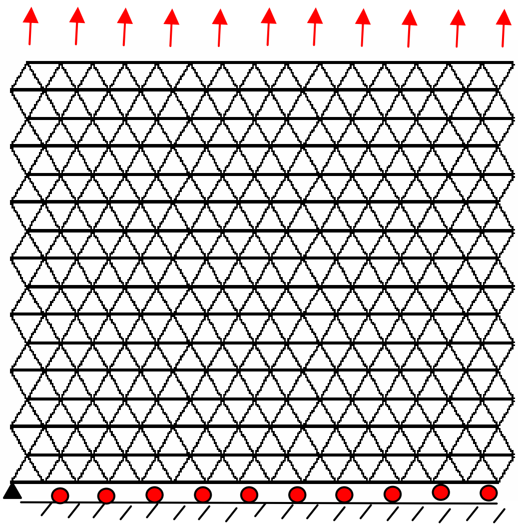

Figure 2. Geometry of the truss lattice model.

3. Lattice parameter calibration

The basic idea in setting up the elastic lattice models is based on the equivalence of strain energy stored in a unit cell (the bold-black part in Figure 4), of a volume V of a lattice with its continuum counterpart (the bold-red part in Figure 4), under uniform strain [9]

$$E_{cell} = E_{continuum} \tag{3}$$

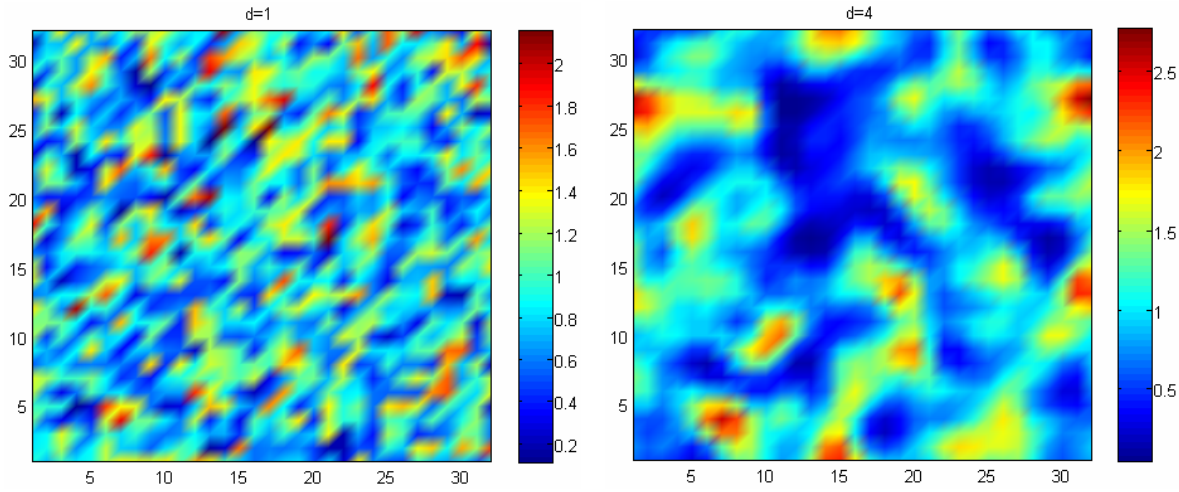

Figure 3. Correlated Weibull RF samples.

The relationship in Eq. (3) is based on the space periodicity of the current regular lattice, which is constructed by arranging the hexagonal unit cell as shown in Figure 4 periodically. The numbers in Figure 4 denote the corresponding nodes. For a regular lattice model, as shown in Figure 2 and Figure 4, the energies of the cell and its continuum equivalent, respectively, are

$$E_{cell} = \frac{1}{2}\sum_{b=1}^{6} \mathbf{F}^{(b)} \cdot \mathbf{u}^{(b)} \tag{4}$$

$$E_{continuum} = \int_V \sigma \cdot \varepsilon dV \Big/ 2 = V\, C_{ijkm}\varepsilon_{ij}\varepsilon_{km} / 2 \tag{5}$$

where, b stands for the b th spring element, C_{ijkm} are material parameters, σ is the stress tensor and ε is the strain tensor. In the two-dimensional (2D) setting, the volume of the unit cell is $V = \sqrt{3} l^2 t / 2$, with t being the thickness of the continuum counterpart of the unit cell and l being the spacing of neighboring unit cells which is equal to the length of the element.

Consider the regular triangular network of Figure 2 with central force interactions only, which are described, for each element *b*, by

$$F_i = k_{ij}^{(b)} u_j = \alpha^{(b)} n_i^{(b)} n_j^{(b)} u_j \tag{6}$$

where $\alpha^{(b)}$ is the spring constant of half-lengths of the central interactions. The unit vectors $\mathbf{n}^{(b)}$ at respective angles $\theta^{(b)}$ of the first three α springs are

$$\theta^{(1)} = 0^\circ,\ n_1^{(1)} = 1,\ n_2^{(1)} = 0$$
$$\theta^{(2)} = 60^\circ,\ n_1^{(2)} = 1/2,\ n_2^{(2)} = \sqrt{3}/2 \tag{7}$$
$$\theta^{(3)} = 120^\circ,\ n_1^{(3)} = -1/2,\ n_2^{(3)} = \sqrt{3}/2$$

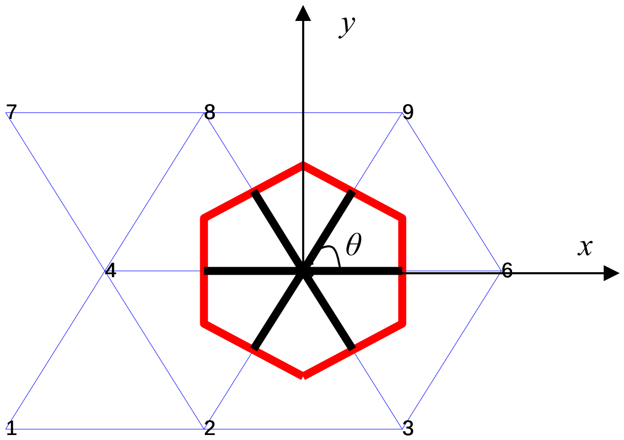

Figure 4. Unit cell of the lattice and coordinate system

Due to the requirement of symmetry with respect to the center of the unit cell, the other three springs ($b = 4,5,6$) have the same properties as $b = 1,2,3$, respectively.
Every node has two degrees of freedom, and it follows that the strain energy of a unit hexagonal cell of such a lattice, under conditions of uniform strain $\boldsymbol{\varepsilon} = (\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{12})$, is

$$E_{cell} = \frac{1}{2}\sum_{b=1}^{6} \mathrm{F}^{(b)} \cdot \mathrm{u}^{(b)} = \frac{l^2}{8}\sum_{b=1}^{6} \alpha^{(b)} n_i^{(b)} n_j^{(b)} n_k^{(b)} n_m^{(b)} \varepsilon_{ij}\varepsilon_{km} \tag{8}$$

From Eqs. (3), (5), and (8), the stiffness tensor can be obtained as

$$C_{ijkm} = \frac{l^2}{4V}\sum_{b=1}^{6} \alpha^{(b)} n_i^{(b)} n_j^{(b)} n_k^{(b)} n_m^{(b)} \tag{9}$$

In particular, taking all $\alpha^{(b)}$ as the same $\alpha^{(b)} = \alpha$, and substituting the value $V = \sqrt{3}l^2 t/2$ and Eq. (7) into (9), we can get

$$C_{1111} = C_{2222} = \frac{9\alpha}{8\sqrt{3}t},\ C_{1122} = C_{2211} = C_{1212} = \frac{3\alpha}{8\sqrt{3}t} \tag{10}$$

It can be observed that the condition

$$C_{1212} = (C_{1111} - C_{1122})/2 \tag{11}$$

is satisfied and there are only two independent elastic moduli, which means the modeled continuum is isotropic. The classical Lame constants can be obtained from Eq. (9) as ($\lambda = C_{1122} = C_{2211}$; $\mu = G = \dfrac{C_{1111} - C_{1122}}{2}$) [18]:

$$\lambda = G = 3\alpha / 8\sqrt{3}t \tag{12}$$

By using the relationship of Young's modulus and the Lame constants

$$E = G(3\lambda + 2G)/(\lambda + G) = 5G/2 \tag{13}$$

$$v = \lambda / [2(\lambda + G)] = 1/4 \tag{14}$$

The equivalent material properties of the spring elements can be expressed in terms of the continuum properties

$$\alpha = 16E\sqrt{3}t/15 \tag{15}$$

Suppose the cross section of the bar element is A, its Young's modulus is E_f and its length is l, we have

$$\alpha = \frac{E_f A}{l/2} = \frac{2E_f A}{l} \tag{16}$$

From Eqs. (15) and (16), we can get the Young's modulus of the bar-element with square cross-section as:

$$E_f = \frac{8\sqrt{3}}{15}\left(\frac{l}{t}\right)E \tag{17}$$

It can be seen from Eq. (17) that Young's modulus of longer fiber elements need to be larger to have the same strain energy as the continuum counterpart, which is in agreement with the conclusion of Liu et al. [7].

4. Numerical algorithm

In this study a random elasto-plastic model is used and the network is constructed from a lattice where all the elements between nearest-neighbor sites are bar elements with the same cross-section area. Finite element method is applied to study the structure response of the lattice network under external tensile loading, as shown in Figure 2. For simplicity, bar element with perfect-plastic properties is chosen, and the perfect plasticity of the bar elements are displayed in Figure 5. The expression of the elemental stiffness matrix is

$$k = \frac{E(\varepsilon)A}{l}\begin{bmatrix} C^2 & CS & -C^2 & -CS \\ CS & S^2 & -CS & -S^2 \\ -C^2 & -CS & C^2 & CS \\ -CS & -S^2 & CS & S^2 \end{bmatrix}, \quad C = \cos\theta, \quad S = \sin\theta \tag{18}$$

where the orientation θ denotes the angle between the axis of the bar element and the coordinate axis, as shown in Figure 4. The Young's modulus of the perfect-plastic bar element $E(\varepsilon)$ can be given as

$$E(\varepsilon)=\begin{cases} E_0 & (0\le\varepsilon\le\varepsilon_Y) \\ \sigma_Y/\varepsilon & (\varepsilon>\varepsilon_Y) \end{cases} \tag{19}$$

where E_0 is the Young's modulus of the bar-element within the elastic limit, σ_Y is the yield stress ε_Y and is the yield strain.

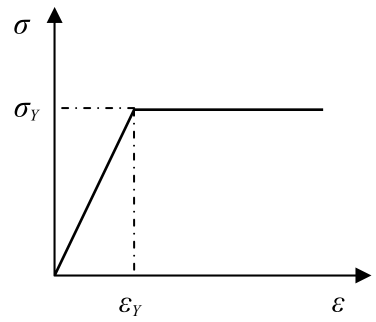

Figure 5. Bar element with perfect plasticity property.

The isotropic damage model with an equal degradation of the elastic moduli postulates the stress-strain law as following

$$\sigma=(1-\eta)E_0\varepsilon \tag{20}$$

$$\eta=1-\varepsilon_Y/\varepsilon,\ (\varepsilon>\varepsilon_Y) \tag{21}$$

where η is a scalar damage variable; for an undamaged material, η is zero and the response is linear elastic. To study the damage process of the lattice network, displacement constraints are imposed on the top and the bottom lines of the lattice system and a secant stiffness algorithm is applied.

The global stiffness of the lattice network can be obtained by assembling all the elements in the network as

$$K=\sum k \tag{22}$$

The network is applied by uniaxial tensile displacement on the top side, as shown in Figure 2, with the left-bottom end being fixed and the y-directional displacement of the nodes on the bottom line being zero. By applying the boundary conditions, we can get a system of non-linear equations to solve for all the degrees of freedom of the system:

$$[K(u)]\{u\}=\{F\} \tag{23}$$

which can be solved by using the modified Newton-Raphson method as

$$\{u_{i+1}\} = \{u_i\} - [[K(u_i)]\{u_i\} - \{F\}]/[K_0] \tag{24}$$

where $[K_0]$ is a secant stiffness matrix, which is kept unchanged within each load cycle, and the subscript i represents the equilibrium iteration. The criterion for stopping the numerical iteration is

$$\|\{u_{i+1}\} - \{u_i\}\|/\|\{u_i\}\| < \delta \tag{25}$$

where $\|\ \|$ denotes a norm and δ is a tolerance value.

5. Numerical example

As paper is a material which has a fibrous network structure consisting of wood fibers, it is convenient to apply the random lattice model to study the failure process of the fibrous structure of paper. Without loss of generality, the Young's modulus of paper can be chosen as $E_c = 2GPa$ [19], and the corresponding Young's modulus of the bar element (within the limit of elasticity) can be obtained using Eq. (17), provided by choosing the length and the cross section of the element as $l = 10mm,\ \ t = 0.1mm$. It is noted that fibers in paper maybe not exactly perfect-plastic as assumed in the present model, the aim of the study is to find the effect of correlation length on global strength when plasticity is considered.

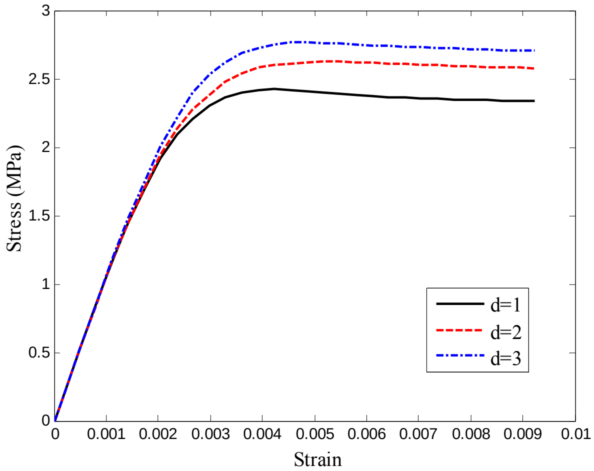

Figure 6. Stress-strain curves of the random elasto-plastic lattice.

The stress-strain curves of the lattices of size (8×8), which denotes the number of nodes of the system is 8×8, are shown in Figure 6. Each curve for a correlation length d is the mean value of stress-strain curves of 100 random lattice samples. The strength of the lattice increases as the correlation length d increases from 1 to 3. It can be seen the phenomena of "strain softening" after the maximum stress value is reached, which corresponds to the decreasing external loads when the stiffness of the system becomes smaller after some elements become plastic. By checking the linear part of the stress-strain curve, we can get the Young's modulus of the lattice network as $E \approx 0.5E_c = 1GPa$, which indicates that the effect of the elasto-plastic properties of the elements will lead to the decrease of the Young's modulus. The macroscopic strength of the fibrous material decreases with reduction of the correlation length, which equivalent to increase of the sample size, this is consistent with the statistical size effect, even when plasticity is presented.

6. Conclusions

In this study a random lattice network model is introduced to simulate the damage behavior of heterogeneous fibrous materials. Elasto-plastic bar elements are used to construct a regular triangular lattice with the random field strength distributions to characterize the randomness of the continuum fibrous materials. The material properties and geometric size of the elements in the lattice networks are calibrated based on the equivalence of the elastic strain energy. Nonlinear finite element method has been applied to study the structural response of the lattice network under external tensile loading. The correlation length of the strength distribution of bar elements has great influence on the strength of the random lattice networks. The macroscopic strength of the fibrous material decreases with reduction of the correlation length, which is equivalent to increase of the sample size.

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