“Sir Alan gave but never broke

Like steel he studied and like oak,

He brought up science by his hand

The royal smith kept on the brand!”

The invariant integral: some news

Genady P. Cherepanov

Honorary Life Member, the New York Academy of Sciences, USA

(Elected on Dec. 8, 1976 together with Linus C. Pauling and George Polya)

genacherepanov@hotmail.com

Abstract. Earlier, this author introduced the invariant integral as a general mathematical tool for solving the physical problems based on conservation laws, without using partial differential equations, similarly to the calculus of variations. In this paper, the invariant integral was introduced for cosmic, gravitational, electromagnetic, and elastic fields combined. In a particular case of the united cosmic-gravitational field, from the corresponding invariant integral the force *F* acting upon point mass *m* from point mass *M* and from the cosmic field was derived : $F = GmMR^{-2} - (4\pi/3)\Lambda GmR$. Here: *G* is the gravitational constant, *Λ is* the cosmological constant, and *R* is the distance between the masses. The first term provides Newton’s gravitation and the second term the cosmic repulsion. This force was used to build an elementary non-relativistic cosmological model of Universe and estimate the size of Universe as well as explain the accelerated expansion of Universe recently observed by astrophysicists. The orbital speed of stars in galaxies was found out to be constant and equal to about 250 *km/s.*

Keywords: invariant integral, cosmic and gravitational field, interaction force, expansion of Universe

1. Introduction

The integrals which are invariant with respect to the integration contour or surface provide a way to write down the laws of conservation of energy, mass, momentum and so on. From them, one can derive the differential equations as the local representation of the same conservation laws. However, the invariant integral approach is more powerful because it allows one to also deal with the field singularities where the differential equations have no meaning.

In 1967, using the energy conservation law, this author derived the main invariant integral for elastic and inelastic materials and introduced it into fracture science [1]. In this approach, the invariance of the integral with respect to any integration paths followed from the energy conservation law, so that this fact seemed to be trivial and was not discussed. Particularly, the characterizing index of power-law hardening materials and the similarity were found out, which constituted the basis of the later HRR approach. In 1968, while studying strain concentration by notches and cracks, Jim Rice utilized this integral to prove its invariance using the divergence theorem. (For this famous work, Jim was awarded several medals and prizes). In 1972, John Landes and Jim Begley, not familiar with paper [1], re-introduced the invariant integral into fracture mechanics. At about that time this author discovered that in the case of elastic materials his integral coincided with that of Jock Eshelby introduced earlier into the theory of point defects in crystal lattices. Since 1968, Jock started on actively working in fracture mechanics, too.

I can't help but recall some events of that time related to paper [1]. Since 1959 my basic scientific interests have been connected with crack growth. However, up to 1964 when I became the youngest Doctor of Science in USSR, my most significant publications were done on the problems of mechanics with unknown boundaries, including the elastic-plastic, local buckling, and contact problems. It's on these problems I earned all my degrees, although fracture has always been my main subject. At that time in USSR, this area was monopolized by unscrupulous, powerful figures close to KGB who vetoed approaches different from their own. Still and all, in 1965 I decided to leave underground and submitted a Russian manuscript entitled "On crack propagation in continuous media" into the journal "Prikladnaia Matematika I Mekhanika" (PMM or Journal of Applied Mathematics and Mechanics, JAMM). However, the publication of the paper was blocked up by the authorities and it was kept in the portfolio of the journal for two years although my earlier, less significant papers used to come out within half a year. And yet, my luck was in because the Editor-in-Chief Leo Galin, even without knowledge of the paper subject, took over the responsibility and published my paper, much mutilated though by censors for two years. At last, it came out by May 1, 1967. Later, I and brave Professor Galin paid a heavy price for this sin. By the way, after the Soviets launched the first sputnik in 1957, the urgent airmail delivered fresh issues of PMM to the best US university libraries within several days.

Since that time, I tried to show that fracture mechanics is a legitimate branch of theoretical physics and my invariant integral can be used as an efficient mathematical tool for solving singular physical problems far beyond fracture mechanics (see my book [2] written in1969 but published in Russian only in 1974 and in English later, in 1978, by McGraw Hill).

However, these ideas were poorly understood. Hopefully, what follows below can make a difference.

2. General case

Let us consider stationary processes in elastic dielectrics, with taking account of cosmic-gravitational and electromagnetic forces. In this case, the main invariant integral can be written as follows [3 - 5]

$$\Gamma_k = \int_\Sigma \left[(8\pi G)^{-1}\left(\varphi_{,i}\varphi_{,i}n_k - 2\varphi_{,i}n_i\varphi_{,k}\right) + \Lambda\varphi n_k + W_0 n_k - D_i n_i E_k - B_i n_i H_k + U n_k - \sigma_{ij} n_j u_{i,k}\, d\Sigma \qquad i, j, k = 1, 2, 3 \qquad (1)$$

$$(W_0 = W + E_i D_i + H_i B_i)$$

Here: Σ, any closed surface in the $Ox_1x_2x_3$ space of Cartesian coordinates; G and Λ, the gravitational and cosmological constants respectively; φ, the potential of united cosmic-gravitational field; E_i, D_i, H_i, B_i, the vector components of electromagnetic field; W and U, the potentials of electromagnetic and elastic fields respectively; u_i and σ_{ij}, elastic displacements and stresses; n_i, the orts of the outer normal to Σ.

Vector $(\Gamma_1, \Gamma_2, \Gamma_3)$ represents the driving force of field singularities inside Σ, which equals the work spent to move the singularities for unit length. If all $\Gamma_k = 0$, then there are no singularities inside Σ; in this case the basic differential equations inside Σ can be derived from the invariant integral (1), e.g. the Maxwell equations, the equations of elasticity theory and gravitation.

Moreover, Eq. (1) allows one to derive the interaction laws for any particular cases, e.g. Newton's law of gravitation, Coulomb's law for electric charges, Ampere's law for electric currents, Joukowski's equation for wing lift, Irwin's law for crack driving force, Peach-Koehler's law for dislocation driving force, Eshelby's law for point inclusion driving force, as well as many new ones [2-5]. For example, the interaction force F of two electric charges q_1 and q_2 moving in a dielectric medium along the common symmetry axis at speed V was found to be equal to [3-5]

$$F = \frac{q_1 q_2}{\varepsilon R^2}\left(\frac{V^2}{a^2} - 1\right)\left(1 - \frac{V^2}{c^2}\right)^{-1} \tag{2}$$

Here: R, the distance between the charges in the proper reference frame; ε, the dielectric constant; c and a, the speed of light in vacuum and medium respectively. For $V = 0$, Eq. (2) represents Coulomb's law. If $V > a$, the force applies only to the rear charge and the force's sign changes. Eq. (2) plays the main part in electron mode fracture [3, 5] by powerful electron beams.

Taking account only of two last terms in the right-hand part of Eq. (1) provides the original invariant integral which is the basis of modern fracture mechanics [1-5].

3. Cosmic-gravitational field

In what follows we consider the united cosmic-gravitational field defined by the invariant integral as follows

$$\Gamma_k = \int_\Sigma \left[(8\pi G)^{-1}\left(\varphi_{,i}\varphi_{,i}n_k - 2\varphi_{,i}n_i\varphi_{,k}\right) + \Lambda\varphi n_k\right]d\Sigma \qquad (i,k = 1,2,3) \tag{3}$$

$$(G = 6.67 \cdot 10^{-11} m^3 kg^{-1} s^{-2},\ \Lambda = 10^{-26} kg/m^3)$$

In Eq. (3), the first term describes the flux of gravitational energy through the closed surface Σ, the second term the work of field tractions on Σ, and the third term the flux of cosmic energy through Σ. In the present non-relativistic approach, the cosmic energy can, probably, be interpreted as “Dark Energy” , but we refrain from using this notion here. By ignoring the third term in Eq. (3) we arrive at the classical model of gravitational field [3-5]. Using physical dualisms the cosmological and gravitational constants can be written in other units.

Let us put $\Gamma_k = 0$ in Eq. (3). It means there are no field singularities inside Σ. In this case, by using the divergence theorem one can easily transform the surface integral in Eq. (3) into the volume integral and derive the following equation for φ which is valid at any point inside Σ :

$$\varphi_{,jj} = -4\pi G\Lambda \qquad (j = 1, 2, 3) \tag{4}$$

From Eq. (4), it follows that Λ represents the density of anti-gravitational matter (negative mass) which is uniformly distributed everywhere. As a matter of fact, this is the physical interpretation of the cosmic field in the present model. For example, in this model the space volume $10^{21} m^3$, which is equal to the volume of our Earth, contains $0.01 gram$ of the anti-gravitational matter so that its density 10^{30} times less than the mean density of Earth.

Now, suppose there is a point mass m at a certain point *O* inside Σ. Using the invariance of the integral in Eq. (3) with respect to Σ , one can shrink Σ and turn it into a small sphere over point *O* . Then, applying the Γ-integration procedure provides the following equation for the force $\boldsymbol{F}(\Gamma_1, \Gamma_2, \Gamma_3)$ upon mass m [3]

$$\Gamma_k = -m\varphi_{,k} \qquad (k = 1, 2, 3) \tag{5}$$

Here, $\varphi_{,k}$ is the kth component of the field intensity vector at point *O,* when there is no mass at point *O.*

The physical nature of the cosmic force as well as gravity remains unclear despite the success of general relativity and numerous other theories.

4. Interaction force

Let us find the interaction law of two point masses in the cosmic-gravitational field. Let mass m_1 be concentrated at point $(0, 0, 0)$. Solving Eq. (4) provides the following field created by this mass

$$\varphi = -Gm_1 r^{-1} - (2\pi/3)G\Lambda r^2 \qquad (r^2 = x_k x_k) \tag{6}$$

The intensity of this field at point $(R, 0, 0)$ is equal to

$$\varphi_{,1} = Gm_1 R^{-2} - (4\pi/3)G\Lambda R, \qquad \varphi_{,2} = 0, \qquad \varphi_{,3} = 0\,. \tag{7}$$

From Eq. (5) and Eq. (7), it follows that force F_2 upon mass m_2 at point $(R, 0, 0)$ directed along the x_1 axis is equal to

$$F_2 = -Gm_1 m_2 R^{-2} + (4\pi/3)m_2 G\Lambda R\,. \tag{8}$$

Also, by analogy force F_1 upon mass m_1 at point $(0, 0, 0)$ is equal to

$$F_1 = Gm_1m_2R^{-2} - (4\pi/3)m_1G\Lambda R. \tag{9}$$

Eqs (8) and (9) provide the interaction law of two point masses in this model. The first term describes Newton's gravitation/attraction, and the second term the cosmic repulsion. The latter does not depend on the opposite mass that plays the only role of a trigger and gauge; and so, the latter does not follow the Newton's law that action equals counteraction. When the distance increases, the gravitation tends to zero while the repulsion tends to infinity.

Let us study some problems for two point masses.

Two free masses in the cosmic-gravitational field. Let free masses m_1 *and* m_2 on the *x*-axis be acted upon by forces of Eq. (8) and Eq. (9) where $R = x_2 - x_1 \geq 0$ (x_1 *and* x_2 are the coordinates of corresponding masses movable along the x- axis). The distance between the masses satisfies the equation

$$\frac{dv}{dt} = -G\left(\frac{m_1+m_2}{R^2} - \frac{8\pi}{3}\Lambda R\right), \quad v = \frac{dR}{dt} \quad . \tag{10}$$

If $3(m_1 + m_2) > 8\pi\Lambda R^3$ at the initial moment of time when $v = 0$, then the masses move one towards the other until they collide. If $3(m_1 + m_2) < 8\pi\Lambda R^3$ at the initial moment of time when $v = 0$, then the masses move apart one from the other until they disconnect. The solution to Eq. (10) is as follows

$$v^2 = 2G\left(\frac{m_1+m_2}{R} + \frac{4\pi}{3}\Lambda R^2\right) + C\,, \quad \text{(where } C \text{ is defined by initial conditions).} \tag{11}$$

One mass is fixed and the other is free. Suppose mass m_1 is fixed at the coordinate origin and mass m_2 is free to move along the *x*-axis. In this case let us take into account the relativistic dependence of the latter mass on its velocity. (Yet, the introduced cosmic-gravitational field does not satisfy the special relativity). In this case the velocity of the free mass can be written as follows

$$ln\left(1 - \frac{v^2}{c^2}\right) = -2Gc^{-2}\left(\frac{m_1}{R} + \frac{2\pi}{3}\Lambda R^2\right) + C\,, \quad \text{(where } C \text{ is defined by initial conditions)} \tag{12}$$

Here *c* is the speed of light.

According to the present model, the cosmic field describes the intrinsic geometrical property of material space for accelerated self-expansion.

5. The cosmological system

Let us apply the introduced cosmic-gravitational field to cosmology.

First, consider a finite system of any number of point masses in a finite volume of the 3D Euclidian space. Designate by L the maximum distance between any two masses, and by M the total mass of the system. The cosmic field attached to this system is inside the sphere of diameter $2L$. According to Eqs (8) and (9) the dimensionless number $Ch = \Lambda L^3/M$

characterizes the ratio of repulsion force to that of attraction and, hence, this number characterizes also the global behavior of this system which depends on its scale. Evidently, when $Ch \ll 1$, we can ignore the cosmic energy and its repulsion effect, and when $Ch \gg 1$, we can ignore the gravitational energy and its attraction effect.

From Eq. (10), it follows that a system of two masses expands and disappears, when $Ch > 3/(8\pi)$, and the system exists, when $Ch < 3/(8\pi)$.

Let us estimate the value of Ch for some astronomical objects.

Our solar system: Sun, Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune, Pluto. The latter is $10^{13}m$ distant from Sun, so that we can take $L = 10^{13}m$ *and* $M = 10^{30}kg$ (about the mass of Sun). From here, it follows that $Ch = 10^{-17}$ for our solar system. And so, there is no way to observe the cosmic field from any planetary observations, although the cosmic field makes the eccentricity of planetary orbits a little bit bigger by an undetectably small amount.

Milky Way. Our galaxy Milky Way consists of more than 200 billion stars which rotate around the galaxy center where there is a Black Hole. Milky Way has a shape of a flat pancake having thickness $10^{19}m$, radius $10^{21}m$, and mass about $M = 10^{42}kg$. From here, it follows that $Ch = 10^{-5}$ for Milky Way. And so, even in the scale of the galaxy it is, probably, impossible to measure the effect of the cosmic field.

Our Universe. Our Universe consists of more than 100 billion galaxies packed in clusters and super-clusters, each of many millions of galaxies. Our Universe's fractal dimension is about 2.2 so that it resembles a flat pancake, too. According to some recent estimates for our Universe, we have $M = 3 \cdot 10^{52}kg$, $L = 3 \cdot 10^{26}m$, and $Ch = 3$.
For typical super-clusters, we have $M = 10^{48}kg$, $L = 4.65 \cdot 10^{24}m$, and $Ch = 1$.

And so, it is evident that the effect of the cosmic field can be observed and estimated only from astronomical observations of very distant objects that are close to the edge of our Universe, i.e. to the time 13.7 billion years since the Bing Bang has happened.

Let us compare the gravitational force of attraction of Earth to Sun and the cosmic force of repulsion of Earth from Sun. The first one is equal to $3.56 \cdot 10^{22}N$ while the second $2.5N$.

Now, we present an elementary, non-relativistic model of our Universe and estimate its size as follows. Suppose our Universe of mass M is homogenous so that a gravitational probe mass m is acted upon by the gravitational force $GmML^{-2}$ of attraction to the center of Universe and by the cosmic force $(4\pi/3)G\Lambda mL$ of repulsion from the center of Universe where L is the distance of the mass from the center of Universe. We define our Universe as a closed community of gravitational masses in the unbounded cosmic field. From here, it follows that the probe mass goes away and leaves Universe if $L > L_\star$ where

$$L_{\star} = \frac{1}{2}\left(\frac{6}{\pi}\frac{M}{\Lambda}\right)^{1/3} \qquad \left(\; Ch = \frac{3}{4\pi} \; \right) \qquad (13)$$

Hence, we can come up with the conclusion that $L_{\star}$ represents the radius of Universe in this model which for $M = 3 \cdot 10^{52} kg$ provides $L_* = 0.71 \cdot 10^{26} m$, i.e. comparatively close to the prediction of the Lambda-CDM model.

If the probe mass is inside the homogenous Universe (i.e. $L < L_*$) , then the total force upon the mass resulting from the gravitational and anti-gravitational matter is zero so that, in average, any domain inside Universe experiences no contraction and no expansion. Moreover, the total mass of Universe is equal to zero. The gravitational matter is concentrated in many moving clots inside Universe while the anti-gravitational matter of the cosmic field is uniformly distributed everywhere. From here, it may be assumed that Universe is a gigantic fluctuation created from nothing (i.e. something which energy-mass was zero).

However, the Universe is open so that, eventually, some gravitational masses on the edge move out and go away, that is they leave Universe forever. Their place in Universe is, then, taken by some insiders. This is the way how Universe expands and, evidently , this expansion is accelerated due to the arising and growing imbalance of gravitational and anti-gravitational forces inside Universe.

The incorporation of the introduced cosmic field into the framework of general relativity seems to be impossible but it is quite plausible for the field theories using non-metric theories of gravity.

6. Orbital speed of stars in spiral galaxies

The orbital speed V of a planet in our solar system determined by equilibrium of inertia force to that of gravitation equals $(GM/R)^{1/2}$ where M is the mass of Sun and R is the distance of the planet from Sun. And so, this speed decreases tending to zero when the distance grows. Because $Ch \ll 1$ in Milky Way and other galaxies, the orbital speed of stars rotating around the center of Milky Way has, seemingly, to be described by a similar law. However, it is not. The orbital speed of stars in galaxies appears to be independent of the distance to the center and equal to about $220 - 260\ km/\mathrm{sec}$ (for our Sun, $225\, km/\mathrm{sec}$).

This paradox produced some theories. For example, the well-known MOND theory accepts that the inertia force upon a star in a galaxy is directly proportional not to the acceleration (equal to V^2/R in the case of uniform circular motion) as follows from the Newton-Galileo mechanics, but to the square of acceleration. Certainly, this approach makes the orbital speed independent of R.

Another school of thought considers Dark Energy responsible for this paradox.

Meanwhile, the structure of Milky Way and other spiral galaxies provides a simple explanation of the paradox using the classical Newton-Galileo mechanics and Newton's law of gravitation which is the right approach because $Ch \ll 1$ in the scale of galaxies so that the cosmic field can be ignored. The gravitational matter of Milky Way is uniformly distributed along the logarithmic spirals having the common pole at the center of Milky Way. These spirals are well-documented. The length of an arc of a logarithmic spiral equals $(R_2 - R_1)/\cos\beta$ where β is the constant angle between the radius-vector of a point on the spiral and the tangent to the spiral at the point while R_1 and R_2 are the distances between the pole and the ends of the arc. The arc length is directly proportional to the distance R of a point from the center of the galaxy when $R = R_2 \gg R_1$. Mass M inside of the galactic disc of radius R is also directly proportional to R because such proportionality remains for any number of spirals. In other words, $M = kR$ where k is a certain structural constant of the galaxy.

The force of attraction of a star of mass m to the center of the galaxy is equal to $Gm \cdot kR/R^2$. This force is balanced by the inertia force of Galileo-Newton. From here, the orbital speed of stars is as follows

$$V = (kG)^{1/2} \tag{14}$$

For Milky Way $k = 10^{21}\ kg/m$ and $V = 250\ km/s$. This value is close to astrophysical data.

About the same value of the orbital speed of stars has been observed in all galaxies. It means that the density of gravitational matter in all galaxies obeys the following general law

$$\rho = k/(2\pi R) \qquad (k = 10^{21}\ kg/m) \tag{15}$$

Here, ρ is the mass of gravitational matter inside unit area of the galactic disc, R is the distance from the center of the galaxy, and k is the universal galactic constant. And so, the matter density is infinite at the center which is the galaxy's Black Hole.

The logarithmic spiral is the only spiral that corresponds to this general law, see Eq. (15).

7. The Einstein equivalence principle and no annihilation paradox

The general relativity is based on the equivalence principle which says that the gravity is the inertial force in the curved space-time. The cosmic-gravitational field does not obey this principle. We illustrate the difference using the following simple example.

Let a mass m , being fixed to a point by an inextensible string of length L , rotate at a constant speed V around the point. In absence of cosmic field, the inertia force mV^2/R is balanced by the extension force in the string. Replacing the action of this string by a gravitational mass M placed at the center of rotation, such that $GmML^{-2} = mV^2/L$, we come up with the equivalence principle. (It can be also formulated as the equity of inertial mass to gravitational one).

Now, imagine a slightly "pliant" string, which cross-section diameter being directly proportional to the square root of the distance from the rotation center and which strain inversely proportional to stress in the string, with a very small proportionality coefficient. In this case, substituting the increased value of L in the above balance equation provides

$$GmML^{-2} - \eta L = mV^2/L \qquad (\eta \text{ is a very small quantity}) \tag{16}$$

Here, L is the undeformed string length which is different from the real one by a small value that cannot be detected in the scale of solar system. In presence of cosmic force, the term ηL imitates the cosmic force and Eq. (16) a modified equivalence principle.

The no annihilation paradox of the present model of cosmic-gravitational field can be understood only after we get known the physical nature of cosmic field. Such a remark is valid also for many non-metric theories of gravity.

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