CIRCULAR RIGID PUNCH WITH ONE SMOOTH AND ANOTHER SHARP ENDS ON AN EDGE CRACKED MATRIX ACTED BY CONCENTRATED FORCES

JUN QIAN and NORIO HASEBE
Department of Civil Engineering, Nagoya Institute of Technology
Gokiso-cho, Showa-ku, Nagoya 466, Japan

ABSTRACT

A vertical circular rigid punch with friction on a semi-infinite plane acted by arbitrarily located concentrated forces with one end smoothly contacting the matrix and another end of sharp corner to initiate an oblique edge crack in the matrix is studied in an analytical way in the present paper. The edge cracked semi-infinite matrix is mapped into a unit circle by a rational mapping function. The problem is classified into the Riemann-Hilbert type, and solved by dividing the whole problem into two parts, one is the edge cracked semi-infinite plane acted by the concentrated forces, the other is the problem solved by substituting the solution of the first part into the R-H equation of the whole problem. The position of the end with corner is fixed while the position of the sliding end is changed with other conditions. After the position of the sliding end is determined by the condition that the stresses at this end are not singular, the stress intensity factors of the crack and resultant moment on the contact region are calculated with different positions of the concentrated forces in the matrix.

KEYWORDS

Circular rigid punch, incomplete contact, oblique edge crack, Riemann-Hilbert problem, rational mapping function, complex function method, Plenelj function

INTRODUCTION

Punch problem is an important branch in contact mechanics. Many classical problems have been solved (Muskhelishvili, 1963, England, 1971, Gladwell, 1980). It is noticed that crack problem is often connected with punch problem owing to the existence of stress concentrations on the contact region and defects in the matrix (Hills and Nowell, 1994). There generally exist two types of contact for a punch with curve shape, one is called complete contact, i.e. the length of the contact region is given, and there exist two corners for the punch coming into the matrix; the other is called incomplete contact, i.e. one or both ends of the punch slide on the matrix, and the length of the contact region is undetermined. When a punch on a matrix with edge defect is referred, some inherent difficulties will appear in computation. Though the problem is possible to be dealt with by some numerical methods such as FEM and BEM, the contact interface and
semi-infinite property of the matrix as well as the singularity of edge crack will result in much inconvenience in computation by FEM, and owing to the lack of the fundamental solution of punch problem, it is not convenient to analyze the problem by BEM. Therefore it is an edge crack, and it is also important to derive the fundamental solution of punch problem with the boundary element method or boundary integral method can be used effectively to analyze the punch on a matrix with edge and inner defects.

A circular rigid punch is complete contact with an edge cracked half plane has been studied (Hasebe and Qian, 1995), and the fundamental solution of the problem has also been derived (Qian and Hasebe, 1996). Since there exists incomplete contact for circular punch with a sharp corner on an edge cracked half plane acted by concentrated forces at an arbitrary point on the interface which is balanced by a horizontal force on the punch. The frictional force is assumed to the punch is usually eccentric to keep the punch vertical, and the eccentric distance of the vertical load on the punch is determined by the resultant moment on the contact region. The position of the magnitude, direction and location of the concentrated forces in the matrix, the frictional material properties of the matrix and radius of curvature of the circular punch as well as the semi-infinite matrix with the edge oblique crack is mapped into a unit circle by a rational by making use of the solution of the semi-infinite plane with the edge oblique crack crack acted by the position of the sliding end of the punch can be decided by satisfying the finite stress condition contact region can be calculated with different positions of the concentrated forces.

PRESENTATION OF THE PROBLEM

As shown in Fig.1, a circular rigid punch is supposed to be vertical on a semi-infinite elastic region. The punch is acted by a vertical load \( P \), and the frictional force on the contact region is coefficient. To keep the punch not to incline, the location of the vertical load on the punch is stresses on the contact region about the origin. The right end with sharp corner has a distance of it is assumed that a pair of concentrated forces acts at point \( z_0 \) in the matrix. To solve the problem in an analytical way, the semi-infinite matrix with the edge oblique crack is mapped into a unit circle by the following rational mapping function (Hasebe and Inohara, 1980):

\[
z = \Phi_0(z) = \frac{1}{1 - \zeta} \zeta + \frac{F_0}{E_0} + \frac{F_1}{E_1} \zeta + \frac{F_2}{E_2} \zeta^2 + \cdots
\]

where \( \Phi_0, F_0, F_1, \) and \( E_0, E_1, \) are known coefficients, and \( E_2, \) is decided by the distance from the edge crack to the punch.

The loading and displacement conditions of the problem can be described as

\[
p_2 = p_2 = 0 \quad \text{on} \quad L = L_1 + L_2
\]

\[
p_2 = -ip_1 = \int p_1 dx = P \quad \text{on} \quad M
\]

\[
\nu = x_1^2 + 2R \quad \text{in the half plane}
\]

\[
\phi(x, y) = (q_1 + iq_2)Q(z, \bar{z}) \quad \text{in the half plane}
\]

\[
L = ABCD, L_1 = IA, M = DE \quad \text{in Fig.1}; \quad p_1 \quad \text{and} \quad p_2 \quad \text{represent the components of traction in x and y directions on the surface of the half plane;} \quad Q(x, y) \quad \text{represents the concentrated forces on the boundary or in the body of the half plane;} \quad \phi(z, \bar{z}) = 1 \quad \text{when} \quad z = z_0 \quad \text{and} \quad 0 \quad \text{when} \quad z = z_0.1 \quad \text{is the displacement of the punch, and} \quad R \quad \text{is the radius of curvature.}
\]

**METHOD OF ANALYSIS**

According to the above loading and displacement conditions, the problem can be transformed into the Riemann-Hilbert problem as follows (Hasebe et al., 1989):

\[
\phi^*(\zeta) + \phi^*(\zeta) = f_{1L} \quad \text{on} \quad L = L_1 + L_2
\]

\[
\phi^*(\zeta) + \frac{1}{\zeta} \phi^*(\zeta) = f_{1M} \quad \text{on} \quad M
\]

where

\[
f_{1L} = \int (p_1 + ip_2) dx
\]

\[
f_{1M} = \int \{ (1 - i\nu)G + (1 + i\nu)(1 + \kappa)R \phi(z) \}
\]

\[
\phi(z) = \phi(z) + \frac{1}{\zeta} \phi(z) + \frac{1}{1 + i\mu} \phi(z)
\]

\[
\frac{1}{(k + 1 - \kappa)(k + 1 - \kappa)} \phi(z) = \phi(z) + \frac{1}{(k + 1 - \kappa)(k + 1 - \kappa)} \phi(z)
\]

\[
R(z) = \phi(z) + \frac{1}{\zeta} \phi(z) + \frac{1}{1 + i\mu} \phi(z)
\]

\[
\psi(z) = \psi(z) + \frac{1}{\zeta} \psi(z) + \frac{1}{1 + i\mu} \psi(z)
\]

where \( \phi(z) \) and \( \psi(z) \) are the complex stress functions of the half plane with the edge oblique
crack acted by the concentrated forces (Hasebe et al., 1996a), and the homomorphic parts of \( \Phi(\zeta) \) and \( \Psi(\zeta) \) are derived by the following procedures. Substituting (4a) into (3a, b), and introducing the Plancherel function \( \chi(\alpha) \), the general solution of \( \Phi(\zeta) \) can be derived as (Hasebe et al., 1989)

\[
\Phi(\zeta) = H(\zeta) + H_1(\zeta) + H_2(\zeta) + \frac{1 + \mu}{2} J(\zeta) + \sum_{\Lambda} \frac{\chi(\alpha)}{\chi(\alpha) - \zeta} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha) - \zeta} \tag{5a}
\]

\[
H(\zeta) = \frac{1}{2\pi i} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{5b}
\]

\[
H_1(\zeta) = \frac{1}{2\pi i} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{5c}
\]

\[
H_2(\zeta) = \frac{1}{2\pi i} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{5d}
\]

\[
J(\zeta) = \sum_{\Lambda} \frac{1}{\chi(\alpha)} \frac{A_0}{\chi(\alpha)} \frac{1}{\chi(\alpha)} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha) - \zeta} \tag{5e}
\]

\[
\chi(\alpha) = (\alpha - \sigma)^m (\sigma - \zeta)^m, \quad m = 0.5 - \ln g / 2g, \quad \rho = (\rho_i + i \rho_o) / (1 + \kappa)
\]

\[
H(\zeta) \text{ is related to the vertical load on the punch. Though it is in integral form, its first}
\]

\[
\text{derive can be expressed in the form without integration (Hasebe et al., 1989).} \quad H_1(\zeta) \text{ is}
\]

\[
\text{related to the displacement on the contact region induced by the radius of curvature of the punch.} \quad \text{Owing to the use of the rational mapping function (1), the integration of}
\]

\[
H(\zeta) \text{ can be carried out explicitly (Hasebe and Qian, 1995).} \quad A_0 \text{ and} \quad A_1 \text{ are determined by solving 2N linear}
\]

\[
\text{simultaneous equations for real and imaginary parts of} \quad A_k = \Phi(\zeta_k) \quad (k = 1, 2, ..., N). \quad H(\zeta) \text{ is}
\]

\[
\text{related to the concentrated forces in the half plane which apply at} \quad \zeta = \zeta_k \text{ and} \quad \zeta = 1 \text{ by the}
\]

\[
F = \log(\sigma - \zeta) - \log(\sigma - 1) + \chi(\alpha) \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{6a}
\]

\[
F_1 = \log(\sigma - \zeta) - \log(\sigma - 1) + \chi(\alpha) \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{6b}
\]

\[
G_i = \frac{\omega(\zeta)}{\omega(\zeta)} \frac{1}{\chi(\alpha)} \frac{1}{\chi(\alpha)} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{6c}
\]

\[
G_2 = \frac{\omega(\zeta)}{\omega(\zeta)} \frac{1}{\chi(\alpha)} \frac{1}{\chi(\alpha)} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{6d}
\]

\[
G_i = \frac{\omega(\zeta)}{\omega(\zeta)} \frac{1}{\chi(\alpha)} \frac{1}{\chi(\alpha)} \int_{\alpha}^{\zeta} \frac{d\alpha}{\chi(\alpha)(\sigma - \zeta)} \tag{6e}
\]

\[
\text{where} \quad A_0 \text{ is decided by the fundamental solution of the half plane with the oblique edge crack}
\]

\[
(1) \text{ by the following expressions:}
\]

\[
\alpha + \rho = \frac{1}{2\pi} \left( \frac{\alpha}{\rho} \left( \frac{\alpha}{\rho} - \beta \right) \right) \tag{7}
\]

\[
\text{Substituting (4a) into (7), it is obtained that}
\]

\[
\alpha + \rho = \frac{1}{2\pi} \left( \frac{\alpha}{\rho} \left( \frac{\alpha}{\rho} - \beta \right) \right) \tag{8}
\]

\[
\text{where} \quad e_0(\alpha), \quad f_0(\alpha) \quad \text{and} \quad g_0(\alpha) \text{ are listed in Appendix I.}
\]

\[
\text{It is found from (8) that the stress components become infinite when} \quad \alpha \text{ tends to} \quad \alpha \text{ or} \quad \beta.
\]

\[
\text{In order to satisfy the finite stress condition at the sliding end} \quad (\alpha - \beta), \text{ the coefficient of} \quad 1 \text{ of} \quad \alpha - \beta \text{ in (8) must be vanished. Therefore the restraining condition is established as follows:}
\]

\[
\frac{1}{2\pi} \left( \frac{\alpha}{\rho} \left( \frac{\alpha}{\rho} - \beta \right) \right) = 0 \tag{9}
\]

\[
\text{If the values of} \quad G_{ii} \text{ and} \quad G_{ii} \text{ are given,} \quad \beta \text{ satisfying (9) can be determined by}
\]

\[
\text{iterative calculation. After} \quad \beta \text{ is determined,} \quad \rho \text{ can be simply decided.}
\]

\[
\text{G}_{ii} \text{ is a parameter which gives the relation among shear modulus} \quad G, \text{ punch width} \quad \rho \text{ (see Fig. 1), radius of the circular punch} \quad R \text{ and vertical force} \quad P, \text{ here} \quad R \text{ is supposed to be larger than the length of the contact region.}
\]

\[
\text{In the following calculation, the concentrated forces are typically assumed to be acted on the}
\]

\[
\text{surface of the half plane in x and y directions,} \quad \kappa = 2, \quad \mu = 0.5, \quad b/a = 0.5, \quad c/a = 0, \quad \gamma = 60^\circ \text{ and} \quad G_{ii} \text{ and} \quad P = 1 \text{ are selected.}
\]

\[
\text{Fig. 2 shows} \quad a' \text{ with different} \quad d, \text{ where} \quad d \text{ denotes the distance from the crack to the}
\]

\[
\text{concentrated force on the surface of the half plane. It is shown that} \quad a' \text{ increases and tends to a}
\]

\[
\text{stable value with the increase of} \quad d.\]
STRESS INTENSITY FACTORS OF THE CRACK

The stress intensity factors of the crack are calculated by

$$K_I = 2\sqrt{\pi} \text{exp}\left(-\frac{\delta}{2}\right) \frac{\Phi' (\alpha)}{\sqrt{\alpha}}$$

where $\delta = (1-2s+i)/(1-2s-i)$ is $\gamma$ on the unit circle corresponding to the tip $C$ of the crack, $s = \gamma / 180$ and $\delta = \gamma / 180$.

The non-dimensional stress intensity factors of the crack are defined as

$$F_I = \frac{K_I}{\sqrt{\pi} \alpha}$$

(10)

Fig. 3 shows $F_I$ and $F_{II}$ with different $d$. $F_I$ decreases with the increase of $d$, and $F_{II}$ is on the contrary. Both $F_I$ and $F_{II}$ tend to stable values which are due to the punch without concentrated force with the increase of $d$.

RESULTANT MOMENT ON THE CONTACT REGION

The problem is analyzed on the condition that the punch does not incline. Thus the resultant moment $R_w$ about the origin of the coordinates can be used to determine the position of the vertical load $P$ on the punch which applies at the distance $e$ away from the y-axis satisfying $Pe = R_w$ (see Fig. 1).

The non-dimensional resultant moment is defined as

$$M_r = \frac{R_w}{\alpha P}$$

where $R_w$ is calculated by

$$R_w = -Re \left[ \int_{\alpha}^{1} \frac{1}{\alpha} d\alpha \Phi'(\alpha) \Phi'(\alpha)/\alpha^2 + \int_{\beta}^{1} \frac{1}{\alpha} d\alpha \Phi'(\alpha)/\alpha^2 \right]$$

(12)

(13)

Fig. 4 shows $M_r$ with different $d$. $M_r$ increases with the increase of $d$ and tends to a stable value which corresponds to the moment of the punch without concentrated forces ($q_1 = q_2 = 0$) in the matrix. The negative value of $M_r$ represents that the moment on the contact region is in clockwise direction with the load $P$ on the right side of y-axis.

CONCLUSIONS

The fundamental solution of circular rigid punch in incomplete contact with a semi-infinite edge cracked matrix acted by a pair of concentrated forces was derived. By simply changing the coefficients of the mapping function (1), the problem with other types of edge defect, such as an edge triangular notch initiating a crack at the tip of the notch (Hasebe and Lida, 1979), can be studied in the same manner. If the radius of the circular punch tend to be infinite value, the solution of the corresponding flat-ended punch can be formed. Since the interface conditions on the contact region, boundary conditions on the surface of the matrix and edge crack are satisfied completely in the process of derivation, the present solution can be forwarded applied to calculate the punch problems with internal defects by boundary integral method or boundary element method without loss of the analytical properties.

APPENDIX I

c_i(\alpha)$,  $f_i(\alpha)$ and  $g_i(\alpha)$ in equation (8)

$$c_i(\alpha) = \frac{1}{4\pi} \int_{\alpha}^{1} (1/\zeta - \alpha)(1/\zeta - \beta)(1 - \alpha)(1 - \beta) \frac{d\zeta}{\chi(1/\zeta - \alpha)(1/\zeta - \beta)}$$

$$+ \frac{1}{4\pi} (\kappa q - q) \frac{(Q_0 - \alpha)(Q_0 - \beta)(1 - \alpha)(1 - \beta)}{\chi(1/\zeta - \alpha)(1/\zeta - \beta)}$$
\[ f(\alpha) = f(\alpha) - \frac{1 - i\mu}{4\pi} q \frac{\omega(\xi)}{\omega(\zeta)} \frac{1}{\omega(\zeta)} \frac{1 - i\mu - \omega(\xi) - \omega(1/\zeta)}{4\pi} \frac{x(\zeta)(\alpha - \xi)}{x(\zeta)(\alpha - \xi)^2} \]

\[ = \frac{1 - i\mu}{4\pi} q \sum_{j=0}^{k-1} \frac{B_{j+1}}{\chi(\zeta)} \frac{1}{\chi(\zeta)} + \frac{1 - i\mu}{4\pi} q \sum_{j=0}^{k-1} \frac{B_j}{\chi(\zeta)} \frac{1}{\chi(\zeta)} \]

\[ g(\alpha) = g(\alpha) + \frac{1 - i\mu}{4\pi} q \frac{\omega(\xi)}{\omega(\zeta)} \frac{1}{\omega(\zeta)} \frac{1 - i\mu - \omega(\xi) - \omega(1/\zeta)}{4\pi} \frac{x(\zeta)(\alpha - \xi)}{x(\zeta)(\alpha - \xi)^2} \]

\[ = \frac{1 - i\mu}{4\pi} q \sum_{j=0}^{k-1} \frac{B_{j+1}}{\chi(\zeta)} \frac{1}{\chi(\zeta)} + \frac{1 - i\mu}{4\pi} q \sum_{j=0}^{k-1} \frac{B_j}{\chi(\zeta)} \frac{1}{\chi(\zeta)} \]

where \( f(\alpha) \) and \( g(\alpha) \) can be found from previous paper (Hasebe and Qian, 1996b)

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