ON STRESS STATE OF FIBROUS COMPOSITE WITH CRACK IN MATRIX

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ABSTRACT

The paper is devoted to analytical investigation of stress-strain state of fibrous composite with matrix crack for the plane and space problems. At infinity composite is subjected to axial stretching coinciding with fibre direction. The crack is perpendicular to load and axis of fibre penetrates through its center. Fibre is treated as unidimensional elastic beam. Its contact with matrix is accomplished along the line with plane problem and on cylindrical area in the case of space problem. In the plane cases the questions of interaction of fibre are touched upon. The asymptotic method is permitting to split the stress-strain state of matrix on components is used. Their determination can be limited by simpler problems.

KEYWORDS

Fibrous, composite, stress state, matrix, crack, analitical method.

INTRODUCTION

Plane problems on influence of strengthening elements on stress state of isotropic plate with crack were considered by Bloom et al. (1966), Cherepanov et al. (1969), Greif et al. (1963), Sanders (1959) by means of analytical and numerical methods. It was shown, that effect of strengthening element was essential, when it was not far from crack end. Maximum decrease of stress near crack top takes place, when crack stretches over strengthening element slightly, thus, study of interaction of strengthening element and symmetric about it crack is of great interest. Such case was considered Pavlenko et al. (1981a) for orthotropic plane. Up to the present time such space problems were practically not investigated, as they contain considerable mathematical difficulties. Besides, the model of unidimensional elastic inclusion in combination with model of contact along line is not applicable immediately in space problems for solids with elastic inclusions, having small cross-sections (Sternberg, 1970).

In this paper the plane and space problems on stress state of fibrous composite with crack in matrix are considered. Composite is stretched along fibres, crack is perpendicular to the load action direction and axis of fibre passes through its center. The questions on interaction of fibres are touched upon in the plane problem. In the space problem the attention is concentrated on one inclusion with constant circular cross-section. It is assumed, that matrix is orthotropic in common case, the main directions of anisotropy coincide either with Dekart coordinate axises $x,y$ or with cylindrical $r,\theta,z$.

ON THE METHOD OF INVESTIGATION

During the investigation of complicated problems of anisotropic and composite materials one
has to deal with the necessity of simplification of corresponding differential equation system. However, perturbation method allow to obtain well-founded approximate equations and to estimate the field of application for different geometrical or physical hypotheses (Mansovich et al., 1982, 1991; Pavlenko 1979, 1980, 1981). The stress-strain state of anisotropic solids is splitted on components having various properties. The determination of the each from the components is reduced to successive decision of boundary problems of potential theory. The two types of components are distinguished in the plane problem for orthotropic solid. If the main axis tension \( \sigma \) and the stress tangential component \( \tau \) play the decisive role and is determined from the equations in the first approximation:

\[
E_u u_{xx} + G u_{xy} = 0 \\
G = E_u u_{yy}, \quad \sigma = G u_y
\]

The displacement \( u \), stress \( \sigma \) and component of tangential stress \( \tau \) are the main components of stress state of second type, they are determined in the first approximation out of the equations:

\[
G u_{xx} + E_2 u_{yy} = 0 \\
\sigma = E_2 u_{yy}, \quad \tau = G u_y
\]

The complete tangential stress \( 
\sigma = E_1 \frac{\partial u}{\partial x} + \frac{\partial \sigma}{\partial y} = E_1 \frac{\partial u}{\partial x} + G \frac{\partial u}{\partial y} = 0
\)

The proposed method allowed to extend essentially the scope of problems, which can be solved analytically. Generalization of the method is carried out by Kagadiy et al. (1992) for viscoelastic material.

**THE PLANE PROBLEM**

Let us consider stress-strain state of fibrous composite with crack in matrix. At infinity composite is subjected to uniaxial stressed with fibrous stretching with intensity \( \sigma_0 \). The crack is perpendicular to load and stretches for equal length on both sides of fibre through its center. Fibre is treated as unidimensional elastic beam. The contact scheme passes along the line. Matrix is orthotropic, the main directions of anisotropy coincide with Dekart coordinate axes \( x, y \). It is assumed, that neighbouring fibres do not influence on stretching along fibre on the infinite orthotropic plate with symmetric crack. Fibre is placed along axis \( x \), crack is placed on the line \( (x, y) = 0 \). The method described above is used for the solution. At the first stage we come to integration of equation (1) with the following boundary conditions:

\[
\sigma = 0 \quad (x, y, z = 0; \quad z = 0) \\
\tau = 0 \quad (x, y, z = 0; \quad x, y, z = 0)
\]

where \( \sigma \) is displacement of fibres. Equation (1) satisfies, if the function \( F \) will be introduced:

\[
\sigma = F, \quad \tau = -F z
\]

Component \( v \) of displacement vector equals zero with \( h = 0 \) (from the conditions of symmetry) then \( v = 0 \) along fibre. Stress in fibre must be \( \sigma / E_1 \), and beam effort \( p \) for cross-section \( x = 0 \) is given by formular (from the requirement of equilibrium)

\[
\sigma = E_1 \frac{A}{E_1} \frac{E_1}{2} + \int \tau \, dz = A \frac{E_1}{E_1} \frac{E_1}{E_1} + 2t \int \tau \, dz = A \frac{E_1}{E_1} \frac{E_1}{E_1} + 2t F
\]

where \( E, A \) are moduls of elasticity and area of fibre cross-section, \( t \) is thickness of matrix. Undimensional variabilities are introduced later.
\[ x = \sqrt{\frac{c}{\epsilon}} + \frac{z}{c}, \quad y = \frac{z}{c}, \quad \frac{\partial u}{\partial z} = \frac{\partial v}{\partial z}, \quad \frac{\partial w}{\partial z} = \frac{\partial v}{\partial z}, \quad \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial z} \frac{\partial v}{\partial z} \]

(9)

we shall consider the field \( y \neq 0 \), as the problem is symmetric. From the equations (1), (7), (9) it follows, that

\[ \Psi_x = \psi_y, \quad \psi_{xy} = -\psi_x \]

Last equalities are the Koshy-Ryan equations, so as function

\[ \Phi(z) = \psi_x + \psi_y \]

is the analytical function of complex variable \( z \). From the equality (8) it follows, that

\[ \psi_y = \mu \psi = P \]

where \( \mu = 2 \pi E \gamma / \gamma \), \( P = P^s E \gamma / \gamma \)

As far as the stress \( \sigma_1 \) is zero on crack, then

\[ \psi = 0 \quad (x = 0, y \neq 0) \]

(11)

From the condition \( \psi = 0 \) in the point \( x = 0, y = 0 \) and on the line \( x = 0, y = 1 \) we obtain:

\[ \psi = 0 \quad (x = 0, y = 0), \quad (x = 0, y = 1) \]

(12)

Thus, the decision of the problem is reduced to definition of analytical in upper semi-surface function \( \Phi(z) \), which meets the conditions (10)-(12). Function \( \Phi(z) \) must meet the additional condition (from the condition at infinity):

\[ \Phi(z) \sim -i z \quad (z \rightarrow \infty) \]

(13)

As a result we obtain

\[ \Phi(z) = -\frac{P}{\mu} - i \sqrt{z^2 + 1} + e^{i \sqrt{z^2 + 1}} \left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

(14)

Under the conditions (11), (12) it follows that \( \Phi(0) = 0 \), \( \Phi(i) = 0 \). It leads to equation system for \( P \) and \( C \), decision of which is as follows:

\[ P = \frac{\pi^2}{2} \frac{B_1 K_1 + B_2 K_2}{B_1 K_1 + B_2 K_2} \int_{c}^{x} \left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

(15)

\[ C = \frac{B_1 K_1 + B_2 K_2}{B_1 K_1 + B_2 K_2} \int_{c}^{x} \left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

(16)

Here \( B_1, B_2 \) are modified Bessel functions, \( L_+(s) \) is a sine function. Function \( \Phi(z) \) (14), displacement \( u \), stresses \( \sigma_1 \) and \( \tau_1 \) are defined by functions \( P \) and \( C \) with formula (9). In particular:

\[ \tau_1 = -\left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

On the line \( x = 0 \)

\[ R_0 \Phi(z) = \omega(y) \quad (y \neq 0) \]

(17)

where

\[ \omega(y) = \frac{c}{2} - \frac{z}{2} - \frac{2\pi}{\mu} \int_{c}^{x} \left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

(18)

From this, tangential stresses do not meet zero boundary conditions on crack. The discrepancy is taken off by the decision of equation (2) under the following boundary conditions

\[ \tau_1 = -\tau_1(x = 0), \quad \nu = 0 \quad (y = 0) \]

All functions appeal to zero at infinity. The decision of the problem (2), (18) is as follows:

\[ \omega(x, y) = \frac{c}{2} - \frac{z}{2} - \frac{2\pi}{\mu} \int_{c}^{x} \left( \frac{2\pi}{\mu} \int_{c}^{x} \frac{1}{y^2 + 1} \right) \]

where

\[ \omega(x, y) = \frac{c}{2} \cos \left( \frac{x}{2} \right) \sin \left( \frac{y}{2} \right) \]

(19)

Now tangential stresses \( \tau \) meets the zero boundary conditions along line \( x = 0 \). Coefficient \( C \) (which is determined from equation (16)) is ratio of coefficients of stress intensity in the crack top for the matrix with fibre and without it. If in cross-section \( x = x_1 \) fibre is torn, then coefficient \( C \) is different from (16) (we shall mark this coefficient \( C^* \)). As \( C^* \), then the advancement of crack in matrix is possible only with broken fibre. If the fibre is not broken, then the question on destruction will be connected with determination of adhesive firm of junction of matrix with fibrous. The influence of neighboring fibres on stress state around crack can be taken into consideration in the following way. Let us assume, that fibres (they are disposed on the distance \( b_1 \) from central fibre) are not displace along axis \( y_j \) with deformation \( y_j \neq 0 \), \( b_1 \) it is problem periodical in direction \( y_j \). That is why we can consider one period and the field \( \psi(x_0, y_j, b_1) \) for symmetric problem. The boundary conditions for the equation (1) are

\[ \psi(x,0,0) = \psi(x_0,0,0), \quad \psi_x(x,0,0) = \psi_x(x_0,0,0) \]

426
The condition of equilibrium of central fibrous leads to equation (10), and of neighbour fibre leads to the equation \( Y_\mu t^\nu = R_\mu t^\nu (y = b) \), \( B = B/L \).

If we introduce function
\[
\frac{f_i(t)}{\delta f_i(t)} + i\eta P(t) + i(P + P_i)
\]
then boundary problem for it may be formulated as follows:
\[
R_\mu t^\nu = 0 \quad (x = 0, \ y = b), \quad J_\mu t^\nu = 0 \quad (y = 0), \quad J_\mu t^\nu = P \quad (x = 0, \ y = b), \quad f_i(t) \sim \mu \quad (x \to \infty).
\]

The decision of this problem may be found by reflection of considered semi-\( z \)-strip to upper semi-surface by means of the function
\[
\xi = \frac{x}{b}, \quad \eta = \frac{y}{b}, \quad \xi = \frac{x}{b}, \quad \eta = \frac{y}{b}
\]
\[
\Phi_\mu(\xi, \eta) = \left( \xi + \left( \frac{\xi^2 + \eta^2 - 1}{2} \right) \right) ^{i\eta b/2b} A_\mu(\xi),
\]
\[
A_\mu(\xi) = \frac{b}{2} \int_0^{\xi} \left( \frac{\xi^2 - 1}{\xi^2 - 1} \right) ^{i\eta b/2b} f_i(t) dt - \left( [P + P_i] / \mu \right) \left( \xi + \left( \frac{\xi^2 - 1}{2} \right) \right) ^{i\eta b/2b}
\]

\( P \) and \( C \) are determined from the condition \( \omega = 0 \) in points \( (x = 0), \ (y = 0), \ P_i \) and \( P_i \) are connected by ratio \( P = P_i = \mu b \).

AXIALSYMMETRIC PROBLEM

Suppose that elastic inclusion (fibre) with circular (radius \( a \)) cross-section is placed in the elastic orthotropic solid (matrix) with cylindrical anisotropy. Axis of fibre coincides with axis \( z \) (cylindrical coordinate system). Cross-section \( x = 0 \) there is disk crack with radius \( \delta \) in matrix. At infinity the solid is subjected to axial stretching by means of stress \( \sigma_0 \) in direction of inclusion. It is necessary to define the distribution law of efforts in fibre, contact efforts between the fibre and matrix, distribution of efforts in solid and influence of fibre on stress state in the crack top. We suppose, that model of unidimensional elastic inclusion continuous in combination with model of contact on cylindrical area for matrix takes place (Pavelenko, 1981; Sternberg, 1970). On the first stage we came to integration of equation (3) under the following boundary conditions
\[
\begin{align*}
\sigma_0 = 0 & \quad (x = 0, \ \eta = 1), \\
\omega = 0 & \quad (x = \infty, \ \eta = 0), \quad \omega = 0, \quad \eta = 1, \quad \omega = w_2 \quad (\eta = 0).
\end{align*}
\]

After the introduction of new variables
\[
\xi = x, \quad \eta = \frac{y}{b}, \quad w = w_1, \quad \xi = x, \quad \eta = \frac{y}{b}, \quad w = w_2
\]
equation (3) will be as follows:
\[
\begin{align*}
\frac{\partial \sigma}{\partial x} + \frac{\partial \omega}{\partial y} = 0
\end{align*}
\]

The last equation is satisfied if function \( F \) is determined from (7) and effort \( P^* \) in

\[
P_i = (AE/E_y) F_y \quad (y = 0), \quad A = 2 \pi a^2
\]

Utilization of ratios (9) (after substitution in (9) \( E_t = E \), \( I = I_n \), \( u = w_1 \), \( t = \xi \) to \( \xi \), \( \xi \) to \( \xi \)) shows, that functions \( p \) and \( \psi \) satisfy the equation
\[
\psi = \varphi_x, \quad - \psi = e^{i\eta b} \psi_0
\]

Equation (21) goes over into equation (10) with \( M = E_k/E \), \( P = E/k \) and the conditions \( P_i \) = 0 on crack and \( \psi = 0 \), when \( \omega = w_2 \), \( \omega = 0 \), \( \eta = 0 \), which satisfy the system (22) and the conditions (10)-(12). Out of the solution decision of system (22) \( \eta = \psi(x, y) \) and \( \psi(x, y) \) we compose the function of the complex variable \( \xi = x + iy \), \( \psi(\xi, \psi) \). The "derivative" \( \psi' \) \( \xi \) is determined in the following way (Lavrentiev et al., 1973)
\[
\psi'(\xi) = \varphi_x + i e^{i\eta b} \varphi_x
\]

Function \( \psi(\xi) \) must satisfy the condition (at infinity \( \varphi < b = \xi \to 0 \))
\[
\psi(\xi) \to -i \xi \quad (\xi \to \infty)
\]

If we introduce function \( f_i(\xi) \)
\[
f_i(\xi) = \Phi(\xi) + i\eta P(\xi) - i\eta P_0(\xi) + i\eta P_0(\xi) = -i(\xi^b \psi_0 \varphi_0 - \omega)
\]

("derivative" \( \phi_0' \) is defined from (23)), then boundary conditions for \( f_i(\xi) \) are as follows:
\[
R_\mu f_i(\xi) = 0 \quad (x = 0, \ y = 1), \quad J_\mu f_i(\xi) = 0 \quad (y = 0), \quad J_\mu f_i(\xi) = P \quad (x = 0, \ y = 1), \quad f_i(\xi) \to i\mu \xi (\xi \to \infty).
\]

Further analysis is analogous with plate problem, as far as the formulas of generalization of Kwood theorem and Kwood formula take place here.

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