A General Damage Criterion for Solids

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ABSTRACT

A generalized damage criterion is introduced which can be reduced to any of a number of known criteria by introducing appropriate values for the material coefficients. In the present paper a simple method for determining an approximate expression for the criterion of craze initiation is also given.

KEYWORDS
Damage Criterion, Yielding, Crazing, Fracturing, Static and Dynamic Loading, Fatigue.

INTRODUCTION

Recently a criterion for time-dependent craze initiation has been proposed (Chern and Hsiao, 1985; Hsiao, 1987). Now this criterion is found to be applicable not only for yielding and crazing but also for fracturing under both static and dynamic loads. Thus this new criterion is a new generalized theory of strength of solids. In the present paper a simple method is also given for determining approximate expressions for craze initiation. Experimental results illustrating the method of obtaining an approximate expression for the craze initiation in polystyrene are given.

GENERAL DAMAGE CRITERION

The general damage criterion may be stated as follows: damage would commence in a solid medium under load if and when the magnitude $S$ of the deviatoric stress tensor overcomes the intrinsic flow resistance $S_0$ in the medium:

$$ S \geq S_0. \quad (1) $$
In terms of the stress tensor components $\sigma_{ij}$ or the principal stresses $\sigma_i$, $S$ may be written in the following form:

$$S = \frac{1}{\sqrt{6}} \left[ (\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 + 6(\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2) \right]^{1/2}$$

$$S = \frac{1}{\sqrt{6}} \left[ (\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 \right]^{1/2}$$

(2)

For different kinds of damage the intrinsic flow resistance of the material may be different. Since $S_0$ is a function of material parameters such as density, microstructure and interatomic or intermolecular forces, the interacting forces are in general associated with the average distance between molecules or atoms. Among many other quantities the specific free volume $\Delta V$ may provide an important dominating clue to the possible structural changes, and may serve as the important parameter to characterize the intrinsic flow resistance $S_0$. Using series expansion $S_0$ may be expressed as follows (Chern and Hsiao, 1985):

$$S_0 = \sum_{n=1}^{\infty} a_n(\Delta V)^n,$$

(3)

where the coefficients $a_n$'s are functions of materials' parameters only.

The specific free volume $\Delta V$, in general, is a function of time $t$, the first invariant $\sigma$ of the stress tensor $\sigma_{ij}$, temperature $T$ and possibly some other quantities such as chemical concentration and the environment.

For linear viscoelastic or elastic media, including most glassy polymers below their glass transition temperature $T_D$, the volume strain or the change in volume per unit volume at a given point which can be obtained by summing the principal strains $\varepsilon_1(x,t)$ or $\varepsilon_2(x,t)$ to give the first invariant of the strain tensor as follows:

$$\Delta V(x,t) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$$

$$= \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}$$

(4)

with zero strains referring to the original unaltered volume at reference temperature $T_D$. Based upon the linear viscoelasticity theory, $\Delta V(x,t)$ can be expressed as the sum of a thermal expansion term and a convolution integral term containing creep compliance $J_2(t)$ and stress rate in diatation $\dot{\sigma}_2(x,t)$ as shown below

$$\Delta V(x,t) = 3a_0 \theta(T) + \int_{-\infty}^{t} J_2(\xi - t) \dot{\sigma}_2(x,t) \, dt,$$

(5)

where $\alpha_0$ is the linear thermal coefficient of expansion, $\theta(t)$ is the temperature function, and $\xi = t \theta(T)$ and $\eta = \alpha_0(T)$ as defined by temperature-time shift principle for thermorheologically simple viscoelastic materials (Schwarzkopf, 1952).

For linear elastic media at reference temperature $T_D$, the following reduction can be found from thermoviscoelastic properties of rheological simple materials

$$\begin{aligned}
\theta(T_D) &= 1 \\
\theta'(T_D) &= 0 \\
J_2(t) &= \frac{1}{3K} \theta(t)
\end{aligned}$$

(6)

with $K$ as the elastic bulk modulus and $\theta(t)$ is the unit step function. Then Eq. (5) can be reduced to:

$$\Delta V(x,t) = \frac{\alpha_0}{3K}.$$  

(7)

which is to be expected as it can be easily obtained from linear elasticity theory directly.

Therefore, for linear viscoelastic media a general criterion for damage initiation may be expressed as (Chern and Hsiao, 1985; Hsiao, 1987):

$$S = \sum_{n=1}^{\infty} a_n \left[ 3a_0 \theta(T) + \int_{-\infty}^{t} J_2(\xi - t) \frac{\partial}{\partial t} \sigma_2(x,t) \, dt \right]^n,$$

(8)

For linear elastic media it may be expressed as:

$$S = \sum_{n=1}^{\infty} a_n \left( \frac{\alpha_0}{3K} \alpha_0 \right)^n.$$

(8)

INITIAL DAMAGE — YIELDING AND CRAZING

(a) Let $a_0 = \sigma_0 \sqrt{3}$, $a_n = 0$ ($n \neq 0)$, where $\sigma_0$ is the yield strength, then the general damage criterion is reducible to

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2.$$

(10)

Eq. (10) is the distortional strain energy criterion put forward by Huber and von Mises long ago.
(b) Let $a_0 = \sigma_y/2$, $a_n = 0$ ($n \neq 0$), under the condition of plane strain with Poisson's ratio $v = 0.5$, the general damage criterion is reducible to the maximum shear stress criterion proposed by Tresca:

$$|\sigma_1 - \sigma_2| = \sigma_y.$$  \hfill (11)

(c) Let $a_0 = \tau_y$, $a_1 = -fK$, $a_n = 0$ (if $n \neq 0,1$), where $\tau_y$ is the yield strength in shear, $f$ is the frictional coefficient, then under the condition of plane strain and $v = 0.5$, the general damage criterion is reducible to Mohr-Coulomb formulation as follows:

$$\tau_{\text{max}} = \tau_y - f\sigma_n,$$  \hfill (12)

where $\sigma_n$ represents the normal stress.

(d) Let $a_1 = B/4K$, $a_0 = A/2$, $a_n = \ell$ (if $n \neq -1,0$), where $A,B$ are material constants, under the condition of plane strain and $v = 0.5$, the general damage criterion is reducible to the form reported by Strumstein and Onghchin (1969):

$$|\sigma_1 - \sigma_2| = A + \frac{B}{\sigma_1 + \sigma_2 + \sigma_3}.$$  \hfill (13)

(e) Let $a_1 = 2X/9K$, $a_0 = 2Y/3$, $a_n = 0$ (if $n \neq -1,0$), where $X,Y$ are material constants, under the condition of plane strain and $v = 0.5$, the general damage criterion is reducible to the form suggested by Oxborough and Bowden (1973):

$$|\sigma_1 - \sigma_2| = \frac{X}{\sigma_1 + \sigma_2 + \sigma_3} + Y.$$  \hfill (14)

(f) Let $a_1 = 0.0156\pi^2d^3/\pi TK$, $a_0 = 0$ (if $n \neq -1$), where $\mu$ is the modulus in shear, $d$ is the molecular diameter displacement, $K$ is Boltzmann's constant, the general damage criterion is reducible to the form similar to that of Argon's (Argon, 1973; Argon and Hannosch, 1977) as shown below:

$$\frac{\sigma_1 + \sigma_2 + \sigma_3}{2\sigma_y} = -\ln(\hat{\nu}_{01}) + \frac{0.00225\pi^2d^3}{KTS},$$  \hfill (15)

with $\hat{\nu}_{01}$ a constant, as the rate increment of porosity.

AN EXPERIMENTAL METHOD OF DETERMINING AN APPROXIMATE EXPRESSION FOR CRAZE INITIATION

Using a sheet specimen of polystyrene containing a central circular hole of radius \( R = 0.03969 \text{ cm} \) (1/64 in) under a tensile stress $\sigma_0 = 48.4 \text{ MPa. (7020 psi)}$, one can apply the following approximate expression for craze initiation

$$S \geq \frac{a_{-1}}{t} + a_0 + a_1 \int_a^t 3\alpha_0 \theta(T) + \int J_2(\xi^{-1}\partial\xi(t,\tau)) \, d\tau \, dt,$$  \hfill (16)

where $\theta$ is the temperature correction to the region around the central circular hole. The boundary equation of the damage area is expressible as follows (Timoshenko and Goodier, 1951):

$$\left[ \left( R^6 - 6(2\cos 2\varphi) \right) \frac{R^6}{r^6} + \left( -1 + 4\cos 2\varphi - 3\cos^2 2\varphi \right) \frac{40}{r^4} \right] + \left( 4 - \frac{10}{3} \cos 2\varphi - 8\cos^2 2\varphi \right) \frac{R^4}{r^2} + \frac{4}{3} \frac{11/2}{r^2}$$

$$= \frac{2a_{-1}}{a_0} \left( \frac{3K}{1 - 2R^2/\pi \cos 2\varphi} \right) + \frac{2a_0}{a_0} + \frac{2a_1}{3K} \left( 1 - 2R^2/\pi \cos 2\varphi \right),$$  \hfill (17)

where $r$ and $\varphi$ are the polar coordinates with the origin in the center of the hole.

Using a Helium-Neon Laser system the picture of the craze area around the hole as shown in Fig. 1 has been obtained.

![](image)

Fig. 1. Damaged Region of Polystyrene Around a Circular Hole under a Vertical Tension as shown.
By choosing several points on the boundary in one of the quadrants of the damaged area, and using data processing methods, the approximate expression for the damage initiation was found to be as below

\[ S \geq \frac{0.8041 \text{ (MPa)}}{1} + 264.6 \text{ (MPa)} \]

\[ 3\alpha_0 \theta(T) + \int_{-\infty}^{t} J_2(\xi - \eta)\delta_{ij}(x, t) \, dt + 26618 \text{ (MPa)} \left[ 3\alpha_0 \theta(T) + \int_{-\infty}^{t} J_2(\xi - \eta)\delta_{ij}(x, t) \, dt \right] \]

(18)

for polystyrene.

In short-time test, the material may be considered as linearly elastic, then Eq. (18) can be reduced to the following form for practical application.

\[ S \geq \frac{8484 \text{ (MPa)}^2}{\sigma_f} - 264.6 \text{ (MPa)} + 2.5\sigma_{ij}. \]

(19)

ULTIMATE DAMAGE — FRACTURE

Now let us return to the general damage expression (9).

\[ \left\{ \begin{array}{l}
 a_0 = \frac{\text{EC}^{1/2}}{1+v} \\
 a_2 = \frac{1}{2} \frac{\sigma_f}{\text{EC}}(3K)^2 \\
 a_4 = \frac{1}{8} \left( \frac{\sigma_f}{\text{EC}} \right)^4 (3K)^4 \\
 a_6 = \frac{1}{16} \left( \frac{\sigma_f}{\text{EC}} \right)^6 (3K)^6 \\
 \vdots \\
 a_n = 0 \text{ (for any other } n) \\
 \end{array} \right. \]

(20)

where \( E \) is the modulus of elasticity and \( C \) is the critical strain energy density, the general damage criterion (9) can be reduced to the following form:

\[ S \geq \left( \frac{\text{EC}}{1+v} \right)^{1/2} \left( \frac{1}{2} ^{2} \frac{1}{6\text{EC}}(\sigma_f)^2 \right) \left( \frac{1}{6\text{EC}}(\sigma_f)^4 \right)^{1/2} \left( \frac{1}{6\text{EC}}(\sigma_f)^6 \right)^{1/8} \]

\[ = \left( \frac{\text{EC}}{1+v} \right)^{1/2} \left[ \frac{1}{6\text{EC}}(\sigma_f)^2 \right]^{1/2} \]

\[ (21) \]

This is precisely the strain energy density criterion for failure under static loading condition (Sih, 1981).

\( (g_2) \) Under certain limitations, the above criterion can be reduced to Sih's strain energy density criterion as a special case as given below:

\[ \frac{1}{2E}(\sigma_f^2 + \sigma_f^2 + \sigma_f^2) \frac{\nu}{E}(\sigma_f^2 + \sigma_f^2 + \sigma_f^2) = C. \]

\[ (22) \]

\( (h_1) \) For fatigue failure, if \( \sigma_f \) in the general damage criterion represent the individual maximum amplitudes of the reversible cyclic stress, then for ductile materials \( \tau_u \leq 0.58\sigma_f \), where \( \tau_u \) is the fatigue limit (shear strength) under reversed torsion stress, \( \sigma_f \) is the fatigue limit (normal stress) under reversed bending stress (Cheng, 1986), and let \( \alpha_0 = \tau_u, \alpha_n = 0 \) (if \( n \neq 0 \)), then the general damage criterion is reducible to

\[ S \geq \tau_u. \]

\[ (23) \]

This is simply Hashin's criterion for ductile materials (Hashin, 1981):

\[ \frac{I_1^2}{\sigma_f^2} - \frac{I_2}{\tau_u} = 1. \]

\[ (24) \]

where

\[ I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{11} + \sigma_{22} + \sigma_{33}, \]

\[ I_2 = \left| \begin{array}{ccc}
 \sigma_{11} & \sigma_{12} & \sigma_{13} \\
 \sigma_{12} & \sigma_{22} & \sigma_{23} \\
 \sigma_{13} & \sigma_{23} & \sigma_{33} \\
 \end{array} \right| + \left| \begin{array}{ccc}
 \sigma_{31} & \sigma_{32} & \sigma_{33} \\
 \sigma_{31} & \sigma_{32} & \sigma_{33} \\
 \sigma_{31} & \sigma_{32} & \sigma_{33} \\
 \end{array} \right| = \sigma_{11} + \sigma_{22} + \sigma_{33}. \]

\[ (h_2) \] For brittle materials, if

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\[ a_0 = \tau_u \]
\[ a_2 = -\frac{1}{2} a_0 \left( 1 - \frac{1}{\sigma_0^2} \right) \left( \frac{1}{3\tau_u^2} \right) (3K)^2 \]
\[ a_4 = -\frac{1}{8} a_0 \left( 1 - \frac{1}{\sigma_0^2} \right) \left( \frac{1}{3\tau_u^2} \right) (3K)^4 \]
\[ a_6 = -\frac{1}{16} a_0 \left( 1 - \frac{1}{\sigma_0^2} \right) \left( \frac{1}{3\tau_u^2} \right) (3K)^6 \]
\[ \cdots \]
\[ a_n = 0 \text{ (for any other } n) \]

then the general damage criterion is reduced to

\[
S \geq \tau_u \left[ 1 - \frac{1}{2} \left( 1 - \frac{1}{\sigma_0^2} \right) (\sigma_0)^2 - \frac{1}{8} \left( 1 - \frac{1}{\sigma_0^2} \right) (\sigma_0)^4 - \frac{1}{16} \left( 1 - \frac{1}{\sigma_0^2} \right) (\sigma_0)^6 - \cdots \right]
\]
\[
= \tau_u \left[ 1 - \frac{1}{2} \left( 1 - \frac{1}{\sigma_0^2} \right) (\sigma_0)^2 \right]^{1/2}
\]  
(26)

which is exactly the criterion (Cheng, 1986; Hashin, 1981) given by Hashin earlier in the following form:

\[
\frac{\sigma_1^2}{\sigma_0^2} + \frac{\sigma_2^2}{\sigma_0^2} = 1.
\]  
(27)

(h4) In the case of biaxial cyclic stresses, i.e., \( \sigma_1 \) and \( \sigma_2 \) are the only nonvanishing stresses, Hashin's criterion is reducible to

\[
\frac{(\sigma_1^2 + \sigma_2^2)}{\tau_u^2} - \frac{1}{\tau_u^2} = 1.
\]  
(28)

which was found to be in good agreement with the experimental data by Rotvel (1970). Eq. (28) has been very useful for analysis and design in pressure vessels.

In fatigue failure research a problem of long standing is the establishment of criteria of failure where the state of cyclic stress is two or three dimensional. In 1981 Hashin derived an isotropic failure criterion for a general three dimensional state of cyclic stress which is claimed to be the most general quadratic approximation to the three dimensional isotropic failure condition for reversed cyclic stresses. However, Hashin's criterion is just a special case of the general damage criterion given in this report. It is seen that the present general damage criterion may well contain and govern the three-dimensional anisotropic nonsymmetrically cyclic stressed conditions.

Apparently this general criterion can give predictive information on various types of damages in spite of the utilization of only one unique free volume concept for the derivation of the criterion.

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REFERENCES


