A New Nonlinear Line-spring Model for Elastic-Plastic Analysis of Surface Cracks

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ABSTRACT
A new nonlinear line-spring model for elastic-plastic analysis of surface cracks is proposed by considering the yielding at back surface. The numerical examples are given and plausible.

KEYWORDS
Surface crack, nonlinear line-spring model, elastic-plastic analysis, back surface yielding.

INTRODUCTION
A nonlinear line-spring model for elastic-plastic analysis of surface cracks based on the D-M model was developed in previous paper (Lu and Jiang, 1983). It does not consider the yielding at back surface of crack. However, this yielding is observed in experiments and has a considerable effect on the yield intensity and opening displacement of the crack tip. Therefore, a new model is proposed by incorporating the yielding at back surface shown in Fig. (a) and (b) in this paper. The nonlinear Constitutive relations of line-spring can be derived from the improved D-M model Solutions for Single edge cracked strip by considering the yielding at back surface, as shown in Fig. 1(c). To model a surface crack in plate or shell, the distributed line-springs are then embedded between the two surfaces of a through crack in plate or shell, as shown in Fig. (d). The solution of the through cracked plate is based on Reissner plate theory. Finally the numerical examples are given.

THE IMPROVED D-M MODEL SOLUTIONS FOR SINGLE EDGE CRACKED STRIPS
The D-M model of a single edge cracked strip can be improved by incorporating the yielding at back surface, as shown in Fig. 1(c). It can be con-
sidered as a double edge cracked strip with yielding stresses acting on the crack surfaces. For this plane strain double edge cracked strip, the equilibrium equations in terms of the displacements are

\[ \begin{align*}
\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + (\psi - 1) \frac{\partial V}{\partial x} &= 0 \\
\frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 V}{\partial x^2} + (\psi - 1) \frac{\partial U}{\partial y} &= 0
\end{align*} \]  

where \( \psi = \frac{2(1-\nu)}{1-2\nu} \) and \( \nu \) is poisson's ratio.

Solving eqns. (1) with the help of the Fourier transform and satisfying the boundary conditions, we get a singular integral equation for the dislocation density function \( \phi(y) \)

\[ \int_\infty^\infty \left[ \frac{1}{\pi} \frac{1}{y^2} + h(y, \xi) \phi(y) \right] dy = \frac{\pi}{\mu} T(\xi) \quad L \in (a, 2a) \cup (a_0, B) \]

where \( \phi(y) = \frac{1}{\pi} \int_\infty^\infty \delta(y - \xi) \phi(\xi) \frac{d\xi}{\mu} \) is the dislocation density function, \( h(y, \xi) = \delta(y - \xi) \) is the thickness function, and the integral is taken in the range of the strip width.

Let \( x = 2(1+1), \ y = a_1 y \) as \( a \) and \( y = a_1 y \) as \( a_1 \) and \( y = a_2 y \) as \( a_2 \), we can write equation (2) as

\[ \int_\infty^\infty \left[ \frac{1}{\pi} \frac{1}{y^2} + h(y, \xi) \phi(y) \right] dy = \frac{\pi}{\mu} T(\xi) \]

where \( \phi(y) = \frac{1}{\pi} \int_\infty^\infty \frac{1}{y^2} \int_\infty^\infty \delta(y - \xi) \phi(\xi) \frac{d\xi}{\mu} \)

\[ \int_\infty^\infty \left[ \frac{1}{\pi} \frac{1}{y^2} + h(y, \xi) \phi(y) \right] dy = \frac{\pi}{\mu} T(\xi) \]

From the strain energy \( U \) of the double edge cracked strip shown in Fig. 1, we have

\[ U = U_0 + \int_a^b dU = U_0 + \int_a^b \left( \frac{2}{E} \frac{d^2 U}{d\xi} + \frac{2}{G} \frac{d^2 V}{d\xi} \right) \]

where \( U \) is the strain energy of the strip without the crack.

Using the relation between the potential energy release rate \( G \) and the stress intensity factor \( K \), we get

\[ G = \frac{1}{2} \int_a^b \left[ \left( \frac{\partial U_0}{\partial \xi} \frac{\partial U_0}{\partial \xi} + \frac{\partial U_0}{\partial \xi} \frac{\partial U_0}{\partial \xi} \right) \right] \]

From relations \( \frac{\partial U_0}{\partial \xi} = \frac{2}{\pi} \) and \( \frac{\partial U_0}{\partial \xi} = \frac{2}{\pi} \), the constitutive relations of line-springs are derived

\[ \int_\infty^\infty \left[ \frac{1}{\pi} \frac{1}{y^2} + h(y, \xi) \phi(y) \right] dy = \frac{\pi}{\mu} T(\xi) \]

where \( \phi(y) = \frac{1}{\pi} \int_\infty^\infty \frac{1}{y^2} \int_\infty^\infty \delta(y - \xi) \phi(\xi) \frac{d\xi}{\mu} \)

The solution of equation (3) is in form \( \phi(y) = (1-\psi)^2 \frac{1}{\pi \xi} \), and \( \phi(y) \) can be numerically obtained from the Lobat-cishebshev method (Theoreanis and Joinam, 1977). The stress intensity factors of the crack tips at \( y = a_1 \) and \( y = a_2 \) are

\[ K_1(a_1) = \int_a^b \left( \frac{\partial U_0}{\partial \xi} \right) \frac{d\xi}{\pi} \]

\[ K_2(a_2) = \int_a^b \left( \frac{\partial U_0}{\partial \xi} \right) \frac{d\xi}{\pi} \]

The fact that the stresses at the crack tips are finite requires \( K_1(a_1) = 0 \) and \( K_1((a_2))^2 = 0 \). We have

\[ G_{\text{SB}}(a_1) = G_{\text{SB}}(a_2) = 0 \]

\[ G_{\text{SB}}(a_1) = G_{\text{SB}}(a_2) = 0 \]

The plastic zone sizes \( \xi_1 \) and \( \xi_2 \) can be obtained from solving eqns. (5) if the crack size \( \xi_1 \) is given.

The crack tip opening displacement is

\[ \xi_r = \frac{2}{\pi} \int_a^b \left( \frac{\partial U_0}{\partial \xi} \right) \frac{d\xi}{\pi} \]

and \( \xi_r = \frac{2}{\pi} \int_a^b \left( \frac{\partial U_0}{\partial \xi} \right) \frac{d\xi}{\pi} \)

where \( \xi_r = \frac{2}{\pi} \int_a^b -1 \) and \( \phi(y) = \frac{1}{\pi \xi} \).
When \( \alpha_0 = 0 \) in Eq. (10), we get the Constitutive relations without the yielding at back surface, and when \( \alpha_1 = 0 \) in Eq. (10), we obtain the Constitutive relations without the crack and with the yielding zone. These Constitutive relations are all necessary to elastic-plastic analysis of surface crack.

When the ligament yielding appear, the Constitutive relation in incremental form of line-spring is given as

\[
\dot{\varepsilon} = \frac{\partial \sigma}{\partial \varepsilon} \dot{\varepsilon} = \{S_{\text{ep}}\} \{d\} 
\]

where

\[
\begin{align*}
S_{\text{ep}} &= \left[ S_0 \right] - \left[ S_{\text{y}} \right] + \left[ \frac{2\beta_1}{\alpha_1} \right] \left[ S_{\text{y}} \right] + \left[ \frac{2\beta_2}{\alpha_2} \right] \left[ S_{\text{y}} \right] + \left[ \frac{2\beta_3}{\alpha_3} \right] \left[ S_{\text{y}} \right] + \left[ \frac{2\beta_4}{\alpha_4} \right] \\
\{d\} &= \{\sigma_0\} \\
\{S_{\text{y}}\} &= \left[ \phi \right] \left[ \phi \right] \left[ \phi \right] \left[ \phi \right] \\
\{\varepsilon\} &= \left[ \varepsilon \right] + \left[ \frac{\partial \varepsilon}{\partial \sigma} \right] \left[ \phi \right] \\
\end{align*}
\]

Substituting the boundary conditions, a system of Cauchy-type singular integral equations can be obtained

\[
\begin{align*}
\int_{-\infty}^{\infty} \frac{\partial G}{\partial x} \frac{\partial \phi}{\partial y} y \, dy &= 0 \\
\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} y \, dy &= 0 \\
\int_{-\infty}^{\infty} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial y} y \, dy &= 0 \\
\end{align*}
\]

where \( \varepsilon / \varepsilon_{\text{cr}} \) and \( \alpha / \alpha_{\text{cr}} \) are the generalised yield function of the edge cracked strip. \( \phi(\sigma_0, \sigma_0, \varepsilon) = 0 \) is the generalised yield surface, and it is obtained from Eq. (5) with \( S_{\text{ep}} = \frac{\partial \sigma}{\partial \varepsilon} \).

To simplify the numerical calculations, the yield surface is linearized. Therefore, \( \partial \phi / \partial \sigma \) is Constant, and the constitutive relation (11) can be integrated into the following form

\[
\{\sigma\} = \left[ \{\sigma_0\} \right] + \left[ S_{\text{ep}} \right] \left( \{\varepsilon\} - \{\varepsilon_0\} \right) 
\]

where \( \{\sigma\} \) and \( \{\varepsilon\} \) are the generalised stress and displacement at which the ligament starts to yield.

**The Behavior Equations of Reissner Plate with a Through Crack**

For Reissner plate theory, we take the basic equations as follows

\[
\begin{align*}
\psi_4 &= 0 \\
\psi_4 &= 0 \\
\psi_4 &= 0 \\
\end{align*}
\]

where \( \nu \) is deflection, \( \sigma \) is stress function, \( \psi \) is additional function and \( \omega = \frac{\partial \psi}{\partial \varepsilon} \).

Having solved Eqs. (13) for \( \nu \), \( \sigma \) and \( \psi \), the stress resultants and couples and the midplane deformations of the plate can be expressed as

\[
\begin{align*}
\beta_\nu &= \frac{3\psi_\nu}{\alpha_\nu} + \frac{3\psi}{\alpha_\nu} \\
\beta_\sigma &= \frac{3\psi_\sigma}{\alpha_\sigma} + \frac{3\psi}{\alpha_\sigma} \\
N_\nu &= \frac{3\nu_\nu}{\alpha_\nu} \\
N_\sigma &= \frac{3\nu_\sigma}{\alpha_\sigma} \\
M_\nu &= \frac{3\nu_\nu}{\alpha_\nu} \\
M_\sigma &= \frac{3\nu_\sigma}{\alpha_\sigma} \\
Q_\nu &= \frac{3\nu_\nu}{\alpha_\nu} \\
Q_\sigma &= \frac{3\nu_\sigma}{\alpha_\sigma} \\
\end{align*}
\]

in which \( \alpha_\nu = 12(1-\mu^2) \).
iterative program, with these solutions, the line-spring generalized forces \( \sigma \) can be obtained from eqs. (10) or eqs. (12), and thus the crack tip opening displacement \( \Delta_c \) can be calculated from eq. (6).

The numerical results of surface cracked plate under tension loadings is given in Fig. 2-5 for \( a/b = 0.2, 0.4, 0.6 \) and 0.8. The curves in these figures give the variations of the dimensionless crack tip opening displacement \( \frac{\Delta_c}{a} \) with tension loading \( \frac{\sigma_0}{a} \) for \( a/b = 0.2, 0.4, 0.6 \) and 0.8. It can be found that the numerical results are plausible.

![Fig. 1](image1)

![Fig. 2](image2)

![Fig. 3](image3)

![Fig. 4](image4)

![Fig. 5](image5)

REFERENCES
