STATISTICAL THEORY OF SIZE EFFECT ON MATERIAL STRENGTH - A REVIEW

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ABSTRACT

A review is conducted to discuss the phenomenon of size effect on material strength; the theory of extreme values is the primary analytical tool used. Weibull (1939) was the first to give a reasonably satisfactory explanation of the volume effect on material strength using the "weakest-link" theory. Experimental data showing how strength of cement mortar decreases with volume are compared with the Weibull theory. The agreement is found to be excellent. Fracture in its general setting, regardless of the mechanics of failure, leads to the same problem, namely, the distribution of the smallest value in large samples enabling the use of extreme value theory. A simple derivation reveals that, regardless of the type of extreme value distribution postulated for material strength (Type I, Type II, and Type III), the mean strength shows dependency upon the size. The size effect of fibrous composites obeys the same general relationship as that for brittle materials. Lastly, the weakest-link concept, and thereby size effect, is directly applicable to fatigue strength at an arbitrarily preassigned life N (cycles) but not to the fatigue life. An attempt is made to summarize the present state of knowledge and to identify unsolved problems requiring further research.

KEYWORDS

Size effect; material strength; statistical theory; brittle fracture; extreme value distribution; material flaw.

INTRODUCTION

Reliability of structures depends strongly on the entering load and resistance distribution functions. Resistance or strength of a material is known to exhibit size effect. By size effect we mean that the strength of a piece of material varies with its dimensions in a way which is typical for the type of material and the geometrical form of the object. The objective of this paper is to study, primarily by reviewing the existing literature, the size effect of materials employing the statistical concepts of material science such as the weakest-link concept or more basic extreme value distributions.

Early Studies Including the Weibull Theory

The phenomenon of size effect on material strength has been known for more than a century. Karmarsch (1859) represented the tensile strength of metal wires by an expression of the form $F = A + B/d$, where d is the diameter and A and B are constants. Chaplin (1880) presented theoretical arguments and experimental data to show that one should expect a decrease in the median (or mean) strength of a bar with an increase in length. Griffith (1920, 1924) has reported the results of theoretical and experimental studies of rupture in glass and other solids, laying the foundation for the "energy theory" of brittle fracture in solids. In order to explain the much greater observed tensile strength of thin wires or fibers as compared with those of larger diameter, Griffith advanced what has become known as the weakest-link or "largest-flaw" concept. Tucker (1927), reporting his studies on concrete columns, concluded that the coefficient of variation (standard deviation/mean) of the compressive strength varies inversely as the square root of the cross sectional area of the specimen. He presents the weakest-link theory, according to which the strength of a column m units long is the strength of the weakest of m individual units, so that the average strength decreases with the length. Harter (1977) offers an extensive discussion of the early literature regarding size effects.

Weibull (1939) presented a statistical theory of the effect of volume on the strength of materials, according to which, the probability of rupture, P_v, at any given distribution of stress, σ, over a volume, v, is defined by the equation

$$\log(1-P_v) = -\int_v n(\sigma)\, dv \tag{1}$$

where $n(\sigma)$ is a function characteristic of each particular material. For statistically homogeneous materials, the material function may be expressed by the formula,

$$n(\sigma) = \left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^\rho \tag{2}$$

where σ_u, σ_o, ρ are constants characteristic of the material (location, scale, and shape parameters, respectively). Combining Eqs. 1 and 2 produces the volume dependent failure probability obtained by Weibull,

$$P_v = 1-\exp\left\{-\left(\frac{\sigma-\sigma_u}{\sigma_o}\right)^\rho v\right\} \tag{3}$$

Size Effect According to Weibull Theory

An explicit relation between the strength and its size may be derived from Eq. 3. Obviously, in estimating the strength, σ, corresponding to any particular P_v, the following relation must be fulfilled if two specimens with different volume v are observed:

$$\left(\frac{\sigma_1-\sigma_u}{\sigma_o}\right)^\rho v_1 = \left(\frac{\sigma_2-\sigma_u}{\sigma_o}\right)^\rho v_2$$

Note that stress and strength are interchangeably used in Eq. 3 because strength is equal to stress at failure. If σ_u may be neglected, a simple expression relating the mean values of strength to the corresponding volume is found:

$$\left(\frac{\sigma_1}{\sigma_2}\right) = \left(\frac{v_2}{v_1}\right)^{1/\rho} \tag{4}$$

In order to illustrate this concept, we briefly report an experiment with hardened cement paste published by Zech and Wittmann (1977). In Fig. 1, the stresses at failure of two different test series are shown as a function of volume v. By applying Eq. 4, ρ can be calculated; values of 14 and 12, respectively, are determined for flexural and tensile tests. The good agreement between the experimental data and the Weibull theory may be considered as an empirical verification of the latter.

STRENGTH CHARACTERIZATION BY OTHER THEORIES

The Weibull theory, based on an empirical assumption such as Eq. 2, is only one of the several theories that explains the size effects that have been observed in static tests and fatigue tests of materials and structures. These other theories are briefly discussed in the following section.

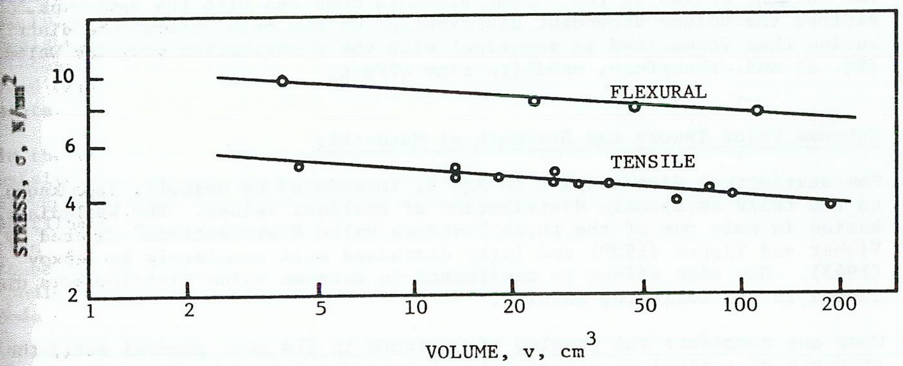


Fig. 1. Flexural and tensile strength of hardened cement paste as function of volume. A solid line is fitted using Eq. 4. (adopted from Zech and Whittmann, 1977).

The Uniform Defect Model

Postulating that the inhomogeneities are uniformly distributed over the volume of a material body, Freudenthal (1968) formulated a fracture model and the probability of failure. The model, which he designated as the uniform defect model, yields a gamma distribution function for fracture strength which shows size effect. This result, although obtained by purely probabilistic reasoning that fracture is caused by a critical number of inhomogeneities, gives rise to the statistical aspect of fracture; that of increasing probability of fracture with increasing volume (or area or length).

The Weakest-Link Concept

The assumption that fracture of the bulk specimen is determined by the local strength of its weakest volume element implies that fracture of the specimen is identified with the unstable propagation of the most severe crack from this element, independently of the local strength of all other elements in the path of the crack. In other words, the fracture process of the specimen is identified with that of a chain, the links of which would be formed by the volume elements; as the strength of the chain is that of its weakest-link, so is the strength of the bulk specimen determined by its weakest-volume element.

Pierce (1926), who was the first to formulate the weakest-link model for fiber strength, was also the first to recognize the close relation of this model to the asymptotic theory of extreme values in large samples of a statistical population. The application of the weakest-link concept to a solid volume rather than to a fiber was first proposed by Weibull (1939) who, however, arrived at the associated distribution function (Eq. 3) by a purely heuristic argument unrelated to the asymptotic theory.

Freudenthal (1968), from the logical assumption of an extreme distribution of the largest cracks in the volume elements combined with the weakest-link, derived the volume dependent distribution of the bulk strength. The distribution thus formalized is identical with the distribution proposed by Weibull (Eq. 3) and, therefore, exhibits size effect.

Extreme Value Theory and Strength of Materials

The statistical distribution in Eq. 3, introduced by Weibull, is also known as the third asymptotic distribution of smallest values. The Weibull distribution is only one of the three "extreme value distributions" discovered by Fisher and Tippet (1928) and later discussed more completely by Gredenkov (1943). How size effect is manifested in extreme value distributions is discussed in the following section.

When one considers the problem of fracture in its most general setting, the strength of a piece of material is clearly determined by its weak points of stochastic strength and location distributed completely at random over the material. From a statistical point of view, this problem leads to the distribution of the smallest value in large samples. Such distributions form the family of extreme value distributions for minima. Using the distribution function of the smallest value, Leadbetter et al. (1983) formally derived the three types of extreme value distributions. Two of these distributions (Type I and Type III), along with their means and standard deviations, are listed in Table 1. The results in Table 1 clearly show that the mean strength m_ℓ is a

function of the size ℓ of the material, a further proof that size effect is a natural phenomenon of materials whose strength distribution is in agreement with one of the extreme value distributions.

TABLE 1 Distributional Parameters Related to Length ℓ (a^{-1} and b are scale and location parameters; γ is Euler's constant; and Γ stands for gamma function). (after Leadbetter et al., 1983)

Initial Distribution	Mean Strength, m_ℓ	Standard Deviation, S_ℓ
Type I: double exponential $F(x) = 1 - \exp(-e^x)$	$b - a^{-1}(\gamma + \log \ell)$	$a^{-1}\dfrac{\pi}{\sqrt{6}}$
Type III: Weibull distribution $F(x) = 1 - \exp(-x^\rho),\ x>0$	$x_o + \ell^{-1/\rho}a^{-1}\Gamma\left(1+\dfrac{1}{\rho}\right)$	$\ell^{-(1/\rho)}a^{-1}\left\{\Gamma\left(1+\dfrac{2}{\rho}\right) - \Gamma^2\left(1+\dfrac{1}{\rho}\right)\right\}^{1/2}$

The "Classical Bundle" Model

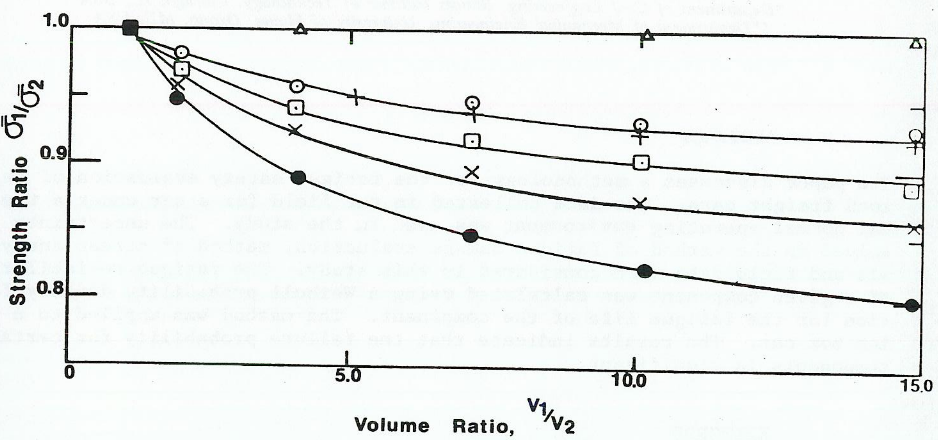
Relaxation of the weakest-link concept, to the extent that instability of the critical flaw size is no longer assumed to lead to fracture of the bulk specimen, gives rise to a material model in which critical crack can be stopped before it propagates from local scale to bulk scale. The fracture strength of this classical bundle model, as well as of the bulk specimen which it is designed to represent, is given by the forces under which a "chain reaction" process of consecutive filament failures resulting from the successive overload carried by the surviving filaments leads to final failure of all filaments.

In the case where the total load is distributed equally over all remaining unfailed parts of the material (equal load sharing rule or ELS), Daniels (1945) showed that the average strength is independent of n (where n is the number of filaments in the bundle) and the variance of strength is an inverse function of n. Freudenthal (1968) noted that the above trends are not usually found in tests leading to brittle fracture, a finding which implies that the bundle model may be more applicable to nonbrittle (ductile) materials which undergo large strain preceding failure than to brittle materials.

Nonbrittle Materials

If a material is (stochastically) nonbrittle, the strength of the whole is not equal to the strength of its weakest part. Different strength models could be proposed here; for example, the stochastic strength model formalized for concrete (Mihashi and Izumi, 1977), the bundle model with either equal load sharing or local load sharing (LLS), whereby the stress is concentrated in the immediate vicinity of failed fibers (Harlow and Phoenix, 1978; Smith, 1980). According to Mihashi and Izumi, the expression of fracture probability for brittle materials is quite similar to that assumed by Weibull yielding identical size effect relationship (see Eq. 4); however, the size effect is less pronounced for nonbrittle materials, as shown in Fig. 2. For purpose of comparison, the size effect relationship of Weibull, LLS, and ELS models is also given in Fig. 2. Suffice it to point out that nonbrittle materials exhibit much less size effect than do brittle solids. This finding is in general agreement with that of Argon (1974). Another interesting result of the Mihashi study is that the scale effect in cases of tensile fracture may be different from that in cases of compressive fracture.

The bundle model, especially the ELS and LLS concepts, is applied primarily in the study of composites wherein bundles such as groups of "filaments" physically exist, a topic which will be discussed in the following section.



●—● Weibull, ρ = 12.0; σ_u = 0; σ_o = 9.2 N/mm^2 (Zech and Wittman, 1977)

×—× Weibull, ρ = 8.0; σ_u = 2.7 N/mm^2; σ_o = 9.2 N/mm^2 (Zech and Whittman, 1977)

⊡—⊡ LLS Rule, ρ = 8.0; k = 3.0; k signifies the number of fiber failures required for bundle failure (Smith, 1980)

⊙—⊙ LLS Rule, ρ = 12.0; k = 3.0 (Smith, 1980)

+—+ Stocastic Model, ρ = 11.0 (Mihashi, 1977)

△—△ ELS Rule, (Gücer and Gurland, 1962)

Fig. 2. Influence of volume on strength.

Composite Materials

We now consider a composite material consisting of high strength, high stiffness, brittle fibers aligned in parallel, embedded in a low-stiffness ductile matrix compound, to which is applied a tensile stress in the axial direction. Harlow and Phoenix (1978) and Smith (1980), employing the LLS concept, concluded that both the bundle and composite are weaker in median strength than

single (short) fiber. An approximate relation conjunctured by Smith is graphed in Fig. 2. The size effect exhibited by this model conforms to the results obtained by Wright and Iannuzzi (1973).

Effect in Fatigue

effect of size on fatigue strength is a complex problem. It frequently depends both upon structural changes in the material and upon the "statistical size effect." Freudenthal (1946) pointed out that the size effect on the initiation stage is opposed by a size effect in the crack propagation stage (at equal rate of propagation, the small cross section will be destroyed rapidly); the resulting size effect depends on the relative magnitude of two opposing effects. For these and other reasons, weakest-link theory not apply in its entirety. As Freudenthal pointed out, the specimen changes with time, and its distribution of strength due to flaws also changes that any essentially static approach which uses the weakest-link concept without modification leaves out certain features of the process. Weibull concluded, however, that the weakest-link concept, initially intended explanation of size effects in brittle materials, is applicable to the fatigue strength of an arbitrarily assigned life N (cycles) but not to the fatigue life. When crack initiation occurs mainly on the surface, the surface area, and not volume, is the appropriate "size" to be taken into consideration. An explicit relation for the ratio between the endurance limits of flawed specimens of cast steel with different stressed volumes (v) has been proposed by Kazinczy (1969):

$$\frac{\sigma_1}{\sigma_2} = \left(\frac{v_2}{v_1}\right)^{1/\rho + n/p} \qquad (5)$$

where ρ is the Weibull shape parameter for the material; n is obtained from size distribution of defects, and p is related to their notch effect. Several Russian researchers have studied the size effect on fatigue strength steel in seawater and other corrosive media; corrosion affects small-diameter specimens more severely than large-diameter ones, so the size effect be opposite in direction to that observed in air.

CONCLUDING REMARKS

statistical theory of the size effect on both static and fatigue strength been well developed; but other theories, especially energy theory, must synthesized with the statistical theory in order to attain a comprehensive theory which will explain more of the existing phenomena. Experimental results on the size effect on both static and fatigue strength of the conventional structural materials (such as metals, wood, and concrete) abound in the literature; but such results for nonbrittle as well as the newer fiber-reinforced composite materials are inadequate.

Weibull distribution, as given by Eq. 3, fits most experimental results brittle materials. From a philosophical point of view, the difficult and doubtful part of Weibull's assumptions is the idea of splitting the body into number of elementary volumes, each of which possesses a strength independent the strength of all other elementary volumes. The idea of elementary volumes can be modified by assuming that the material consists of randomly distributed flaws. Fracture may develop from such a flaw and the stress which causes a flaw to spread fracture may be independent of the corresponding stresses for all other flaws in the body. This concept merits consideration in future studies.

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