NUMERICAL APPLICATIONS OF PATH INDEPENDENT INTEGRALS IN THE CASE OF THERMAL STRAINS, CREEP ANALYSIS AND MIXED MODE SITUATIONS

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ABSTRACT
Numerical determination of J-Integral by finite element path integration technique can be extended and applied to situations such as thermal strains (area term), steady state creep (C*-Integral evaluated from velocity fields), mixed modes in linear elastic fracture mechanics (J_I, J_{II}). Path independent quantities are calculated and can be used to correlate crack initiation or growth phenomena in these situations.

KEYWORDS
Fracture mechanics; finite element analysis; J-Integral; path independence; thermal strains; creep; C*-Integral; mixed mode; (J_I, J_{II}) integrals.

INTRODUCTION
This study is concerned with two-dimensional finite element determinations of J-Integral and extensions in various situations. After a mechanical analysis performed with INCA code (CASTEM system), a post processor is used to calculate line and area integrals. Three situations are considered: a longitudinally cracked tube subjected to thermal loading in the elastoplastic regime (J determination); a centre cracked plate made of a creeping material under uniform tension (C* determination); a slant edge crack in a plate subjected to tension or bending in the elastic regime (J_I, J_{II} determination).
EVALUATION OF J-INTEGRAL IN THE PRESENCE OF A TEMPERATURE FIELD

If we consider the case of 2D fracture mechanics analysis in the presence of a thermal strain field:

\[ \varepsilon_{ij}^{th} = a (\Theta - \Theta_R) \delta_{ij} \]

- \( a \) : dilatation coefficient
- \( \Theta \) : temperature field
- \( \Theta_R \) : mean temperature

It is easy to see that the usual expression of \( J \) loses its fundamental path independence property, when evaluated with the mechanical strain \( \varepsilon_{ij}^{m} \) in the energy density term:

\[
J = \int_{\Gamma} \left( w^{m} d y - T^{m} \frac{\partial u^{m}}{\partial x} d x - d l \right)
\]

\[
w^{m} = \int_{\Omega} \sigma_{ij}^{m} d \varepsilon_{ij}^{m}
\]

(1)

In order to re-establish path independence it appears necessary to add an area integral to the previous line integral (Bui, 1978):

\[
J = J + \iint_{A} \frac{\partial \Theta}{\partial x} d x d y
\]

(2)

- \( A \) : surface surrounded by the contour

This expression is valid for a material which is temperature independent; if the properties (Young's modulus, stress-strain curve, dilatation coefficient) are strongly temperature dependent, relation (2) can be generalized in the following form:

\[
J = J + \iint_{A} \left[ a + (\Theta - \Theta_R) \frac{d a}{d \Theta} \right] \sigma_{ii}^{m} - \frac{W_{el}}{E} \frac{d \varepsilon_{ii}^{m}}{d \Theta} - \int_{\Omega} \frac{d \varepsilon_{ii}^{m}}{d \Theta} \frac{d \Theta}{d x} d A
\]

(3)

\( W_{el} \) : elastic part of mechanical strain energy density

Numerical application was conducted under temperature independent material assumptions, which already allows a large number of industrial applications. The finite element analysis of a longitudinally cracked tube (\( R_1 = 0.05 \) m; \( a/t = 0.5 \); \( t/R_1 = 0.2 \)) was performed with isoparametric 6-noded triangular and 8-noded quadrangular elements in plane strain situation.

Material properties are:

- \( E = 200 \) GPa
- \( \nu = 0.3 \)
- \( a = 10^{-5} \) K⁻¹

Stress-strain law \( (\sigma > \sigma_o = 200 \) MPa\):

\[ \frac{\sigma}{\sigma_o} = \left( \frac{\varepsilon}{\varepsilon_o} \right)^5 \]

Temperature field is given by a logarithmic radial distribution (\( R_1 < r < R_o \)):

\[ \Theta(r) = \Theta_o \frac{\log(r/R_1)}{\log(R_o/R_1)} \]

The numerical evaluation of relations (1) (2) is carried out in the following way: quantities such as stresses or displacement gradients are determined at each nodal point of the contour; line integral is evaluated by using linear interpolation between two consecutive nodes and area term is evaluated by Gauss point technique. A linear elastic calculation was carried out for \( a \Theta_o = 0.015 \). Results thus obtained are plotted on Fig. 1 where the importance of the surface integral is clearly seen (contour radius corresponds to a circle of the same area). The selected value can be chosen as \( J = 410 \) kNm⁻¹. In the thermoplastic regime analysis was conducted in seven regular temperature steps from \( \Theta_o = 30 \) K to \( \Theta_o = 210 \) K; incremental plasticity equations based on Von Mises criterion and normality rule are solved by using a two level iteration scheme of initial stress type. Numerical results are given in Table 1 and plotted on Fig. 2 for the last step; one can notice that path independence is well maintained. The contribution of the area term can reach 30% of the total value for the largest contour at the last step; the influence of plasticity on J-Integral is moderate in that case (see ratio \( J/J_{el} \) in Table 1).

<table>
<thead>
<tr>
<th>( \Theta_o ) (K)</th>
<th>( J ) (kNm⁻¹)</th>
<th>( J/J_{el} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>0.16</td>
<td>1.00</td>
</tr>
<tr>
<td>60</td>
<td>0.67</td>
<td>1.01</td>
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<tr>
<td>90</td>
<td>1.51</td>
<td>1.10</td>
</tr>
<tr>
<td>120</td>
<td>2.27</td>
<td>1.25</td>
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<tr>
<td>150</td>
<td>5.41</td>
<td>1.32</td>
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<tr>
<td>180</td>
<td>8.19</td>
<td>1.39</td>
</tr>
<tr>
<td>210</td>
<td>11.4</td>
<td>1.42</td>
</tr>
</tbody>
</table>

Table 1: Evaluation of \( J \) in thermoplasticity

![Fig. 1 Thermoeelastic results](image1)

![Fig. 2 Thermoplastic results](image2)
This small study shows the possible use of a criterion based on $J$ to predict crack initiation in the presence of thermal strains from a finite element point of view. It can be applied for instance in the important case of a flawed structure subjected to a thermal shock and material toughness $\tilde{\lambda}_{1}$ corresponding to the temperature around the crack tip must be used.

**NUMERICAL EVALUATION OF C*-INTEGRAL FOR A CREEPING MATERIAL**

For a creeping material $J$ can no longer be considered as a crack tip field parameter because it loses its fundamental path independence property. Begley and Landes (1976) introduced the $C^*$-Integral as a crack tip field velocity parameter:

\[ C^* = \int_{C^*} (W^* - \frac{\partial T}{\partial \gamma} - \frac{\partial F}{\partial u} - du) \]

\[ W^* = \int_{C^*} \frac{\partial u}{\partial t} \, dt \]

(4)

In the usual case of a Norton law $\dot{\epsilon} = A \sigma^N$, $W^*$ reduces to:

\[ W^* = \frac{N}{N + 1} A \sigma^{N + 1} \]

(5)

Dang-Van and Mudry (1981) demonstrated path independence of $C^*$-Integral in the case of steady state creep ($\dot{\gamma} = 0$) for a non-moving crack. Numerical application was done on a centre cracked plate (crack length $2a = 0.1$ m; height $2H = 0.5$ m; width $2w = 0.2$ m) under uniform tension $\sigma_{np}$ (see mesh on Fig. 3). Material properties are:

\[ E = 200 \text{ GPa} \]

\[ \nu = 0.3 \]

\[ \text{Stress strain law } \epsilon = (\frac{\sigma}{\sigma_0})^{13} \text{ with } \sigma_0 = 400 \text{ MPa} \]

\[ \text{Creep law } \dot{\gamma} = A \sigma^N \text{ with } N = 8 \text{ and } A = 10^{-24} \text{h}^{1}\text{MPa}^{-8} \]

Finite element analysis in plane strain situation was divided into two phases: elastic-plastic loading during which stress is gradually increased from $\sigma_{np} = 20$ MPa to $\sigma_{np} = 200$ MPa; creep phase where load is kept constant ($\sigma_{np} = 200$ MPa) during three hours. In relation (4) displacement velocity $\dot{u}$ is evaluated by finite difference of displacement fields for two consecutive time steps: $\dot{u} = \Delta u / \Delta t$.

The following results have been obtained: in the early stage of creep $C^*$ presents a strong path dependence but this dependence tends to vanish when a steady state is reached (see Fig. 4) and $C^*$ becomes constant ($C^* = 29$ kN m$^{-1}$ h$^{-1}$).

**Fig. 3** Quarter of a centre cracked plate

**Fig. 4** $C^*$ path independence

A simple correlation is used by experimentalists to estimate $C^*$ from crack mouth opening rate measurements in the case of a CCP specimen (Taira and others, 1979):

\[ C^* = \frac{N - 1}{N + 1} \frac{\sigma_{\text{net}}}{\sigma_{\text{avg}}} \Delta \]

(6)

$\sigma_{\text{net}}$: averaged stress in the uncracked ligament

Our finite element results lead to $\dot{\gamma} = 0.074$ mm h$^{-1}$ which, by application of relation (6) gives $C^* = 30$ kN m$^{-1}$ h$^{-1}$. This result is very close to the path integration result. $C^*$-Integral can also be evaluated with EPRI handbook estimation technique (Kumar and others, 1981) through the following relationship deduced by analogy with fully plastic $J$ formula.

\[ C^* = A_1 (1 + \frac{a}{W}) \frac{1}{N + 1} \left( \frac{\sigma_{\text{net}}}{2 \sqrt{3}} \right) \]

(7)

In our case $A_1 = 1.68$ and the application of relation (7) leads to $C^* = 32$ kN m$^{-1}$ h$^{-1}$. As the EPRI estimation procedure cannot take into account effects of plasticity prior to creep phase, we consider that there is a good agreement between that value and our finite element results.
This study shows the numerical applicability of $C^*$-Integral concept to fracture mechanics creep problems in the case of a stationary crack when steady state is reached. The results support a linear correlation between $C^*$ and the crack opening rate $\delta$; a good consistency was found with EPRI estimation scheme. Practical use of $C^*$ values can be made through experimentally determined laws of the following form:

$$a = f(C^*)$$

**NUMERICAL EVALUATION OF ($J_I$, $J_{II}$) INTEGRALS IN A MIXED MODE SITUATION**

In the case of a 2-D mixed mode problem the use of J-Integral only does not provide a method for determining separately stress intensity factors because $J$ is a quadratic sum of $K_I$ and $K_{II}$. A modified formulation introduced by Bui (1982) now permits this decoupling.

If we consider two symmetrical part points $M$ and $M'$ with respect to the axis of the crack, it is possible to construct the symmetrical part (mode I) and antisymmetrical (mode II) of the displacement fields:

$$\begin{align*}
\{u_I\} &= \frac{1}{2} (u_M + u_{M'}) \\
\{v_I\} &= \frac{1}{2} (v_M - v_{M'}) \\
\{u_{II}\} &= \frac{1}{2} (u_M - u_{M'}) \\
\{v_{II}\} &= \frac{1}{2} (v_M + v_{M'})
\end{align*}$$

(8)

Associated elastic strain and stress fields must be separated in the same way. In these conditions the two following line integrals can be defined:

$$J_I = \int_T (W (\sigma : \varepsilon) \; d\gamma - T : \varepsilon) \cdot \frac{\partial u_I}{\partial x} \; dl$$

$$J_{II} = \int_T (W (\sigma : \varepsilon) \; d\gamma - T : \varepsilon) \cdot \frac{\partial u_{II}}{\partial x} \; dl$$

(9)

These integrals are shown to be path independent and uniquely related to the corresponding stress intensity factors:

$$J_I = \frac{K_I^2}{E' \sigma} \quad J_{II} = \frac{K_{II}^2}{E' \sigma} \quad J = J_I + J_{II}$$

(10)

Elastic finite element analysis was carried out on a plate with a slant edge crack in plane strain situation (see mesh and geometry on Fig. 5).

Material properties are:

$$E = 200 \text{ GPa} \quad \nu = 0.3$$

Two types of loading are considered: uniform tension $\sigma = 100$ MPa and bending moment $M$ produced by a linearly varying stress of magnitude $\sigma_b = 6M/b^2 = 100$ MPa.

Line integrals ($J_I$, $J_{II}$) are evaluated on six circular contours surrounding the crack tip; results concerning tension loading are plotted on Fig. 6. Numerical values are given in Table 2 as the well as corresponding stress factors:

$$F_I = \frac{K_I}{\sigma \sqrt{\pi a}} \quad F_{II} = \frac{K_{II}}{\sigma \sqrt{\pi a}}$$

These results are in good agreement with those published in the compendium of stress intensity factors (Rooke and Cartwright, 1976).

**Table 2 Mixed mode results**

<table>
<thead>
<tr>
<th></th>
<th>$J_I$ (kNm$^{-1}$)</th>
<th>$J_{II}$ (kNm$^{-1}$)</th>
<th>$F_I$</th>
<th>$F_{II}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tension</td>
<td>0.980</td>
<td>0.239</td>
<td>1.17</td>
<td>0.57</td>
</tr>
<tr>
<td>Bending</td>
<td>0.425</td>
<td>0.072</td>
<td>0.77</td>
<td>0.32</td>
</tr>
</tbody>
</table>

**Fig. 5 Slant edge crack in a plate**

This major interest of the previous method for determining $K_I$ and $K_{II}$ is to take advantage of path independence property of $J_I$ and $J_{II}$, which does not necessitate a very refined mesh in the vicinity of the crack tip. Moreover it is easy to implement in an already existing J-Integral evaluation code because it uses the same integration routines; it can also be generalized to thermal problems.

It is often important to be able to estimate $K_I$ and $K_{II}$ separately in many practical cases such as directional criteria, mixed mode fatigue crack growth,...
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REFERENCES