SOME PROBLEMS IN THE J-INTEGRAL

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ABSTRACT

For a blunt crack the J-integral is path dependent on contours very closed to the crack tip even for elastic material. Using the incremental J-integral theory we introduce a new parameter J_t charaterizing the behavior of a crack tip and prove that the J-integral is nearly path independent on contours whose radii are greater than several COD if $\sigma_{ij,1}\Delta\varepsilon_{ij} - \varepsilon_{ij,1}\Delta\sigma_{ij} = 0$ in the plastic regions.

KEYWORDS

Blunt crack; ideal crack; J-integral; incremental J-integral; finite deformation; updated Lagrange method; Lagrange stress components; Euler stress components.

INTRODUCTION

In the present time it is proved that for the ideal crack the J-integral introduced by Eshelby, Rice and Cherepanov is path independent in the cases of linear elasticity and power hardening plasticity within the context of deformation theory of plasticity. But for more realistic incremental theory and the blunt crack is not solved now(Atluri,1977; Mcmeeking,1977; Miyamoto, 1981) In this paper we will dicuss this problem in general for the small strain and get some useful results. For the finite strain we will discuss shortly and results are similar.

ELASTIC BLUNT CRACK

Let O be the general focus of the blunt crack. Let O also be the original of the local coordinate system. A is the crack tip. $OA=R_o$ is the general focal length (Fig. 1). The stresses near the blunt crack end for the mode I are (Kuang, 1982)

$$\sigma_{11} = (K_1 /\sqrt{2\pi r})(\cos\tfrac{\theta}{2} - \tfrac{1}{2}\sin\theta \sin\tfrac{3\theta}{2} - \tfrac{R_o}{r}\cos\tfrac{3\theta}{2})$$
$$\sigma_{22} = (K_1 /\sqrt{2\pi r})(\cos\tfrac{\theta}{2} + \tfrac{1}{2}\sin\theta \sin\tfrac{3\theta}{2} + \tfrac{R_o}{r}\cos\tfrac{3\theta}{2}) \qquad (1)$$
$$\sigma_{12} = (K_1 /\sqrt{2\pi r})(\tfrac{1}{2}\sin\theta \cos\tfrac{3\theta}{2} - \tfrac{R_o}{r}\sin\tfrac{3\theta}{2})$$

According to the definition of the J-integral we have

$$J = \int_{\Gamma} (W\, dx_2 - \vec{T}\,\vec{u}_{,1}\, ds) \qquad (2)$$

where Γ is a counterclockwise path encircling the blunt crack end. For simplicity we take Γ as a circle arc with radius R_1 and its center is located at O. $\vec{u}_{,1} = \partial\vec{u}/\partial x_1$, ds is a differential arc length along the Γ. Other symbols are the usual notations. For the plane problems we have

$$W = \frac{1}{2E'}\{ \sigma_{11}^2 + \sigma_{22}^2 - 2\nu'\sigma_{11}\,\sigma_{22} + 2(1+\nu')\,\sigma_{12}^2 \} \qquad (3)$$

$$T_i = \sigma_{ij}\, n_j \,, \quad n_1 = \cos\theta \quad n_2 = \sin\theta \qquad (4)$$

where for plane stress: $E' = E, \quad \nu' = \nu$

for plane strain: $E' = E /(1 - \nu^2), \quad \nu' = \nu/(1 - \nu)$

Substituting (1) into (3) and $\vec{T}\cdot\vec{u}_{,1}$, we get

$$W = (K_1^2/2\pi r E') \cos^2\tfrac{\theta}{2}\{(1 - \nu') + (1 + \nu') \sin^2\tfrac{\theta}{2}\} \qquad (5)$$

$$\vec{T}\cdot\vec{u}_{,1} = (K_1^2/8\pi r E')\{ -(3 + \nu') + (9 - 7\nu')\cos\theta/4 + (5 - \nu')\cos 2\theta - (1+\nu')\cos 3\theta/4 \} \qquad (6)$$

Substituting (5), (6) into (2) we get

$$J = (K_1^2/4\pi E')\{ 4\varphi + (1+\nu')\sin\varphi/4 - 2\sin 2\varphi - (1 + \nu')\sin 3\varphi/12 \} \qquad (7)$$

where φ is the polar angle at the end point of Γ (Fig. 1). We may approximately take the equation of the end of a blunt crack as (Kuang, 1982)

$$R = R_o / \cos^2\tfrac{\theta}{2} \qquad (8)$$

So φ can be determined as

$$\cos(\varphi/2) = \sqrt{R_o/R_1}$$

The calculated result shows that the effect of ν' is small. Table 1 and Fig.2 show the relationship between $R_1/(2R_o)$ and $J/(K_1^2/E')$

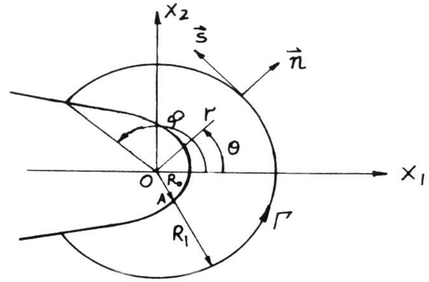

Fig.1

Table 1 ($\nu' = 0.4$)

$R_1/(2R_o)$	1/2	4/7	2/3	1	2	4	8	15
φ	0°	41.4°	60°	90°	120°	138.6°	151°	158.9°
$J/(K_1^2/E')$	0	0.08	0.22	0.536	0.829	0.935	0.977	0.991

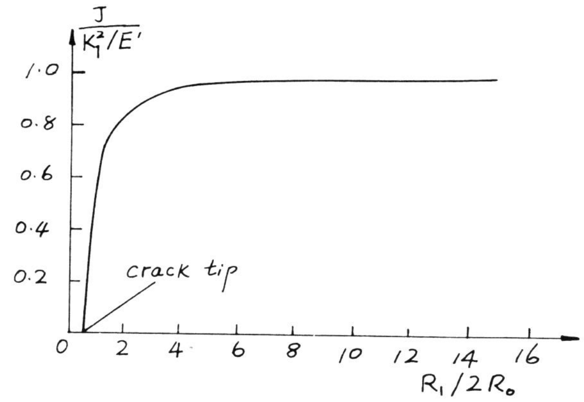

Fig.2

Comparing Fig.2 with Mcmeeking's result(Mcmeeking,1977) we find that the relationship between $R_1/(2R_o)$and $J/(K_1^2/E')$ are similar in spite of the difference in methods and constitutive equations. From Fig.2 we know that the J-integral is path dependent when $R_1/(2R_o) < 6{\sim}8$ for the blunt crack. If we account the variations of geometrical configuration for a initial ideal crack in the calculating process then the above results also tenable. Therefore the most possible main source of J's path dependence is that the singular point O don't be entirely enclosed by Γ. But the J-integral is identical for any two Γ_1, Γ_2, if their initial and end points at the crack boundary are the same. the reason is obvious.

INCREMENTAL J-INTEGRAL IN NONLINEAR ELASTICITY

Let the incremental J-integral be

$$\Delta J = \int_\Gamma \{\Delta W\, n_1 - \Delta(\,T_i\, u_{i,1}\,)\}\, ds \qquad (10)$$

where ΔW, $\Delta(\,T_i\, u_{i,1}\,)$ are the incremental W and $(\,T_i\, u_{i,1})$ from a deformation state M to M + 1. For nonlinear elasticity we have

$$W = W(\,\varepsilon_{ij}), \qquad \sigma_{ij} = \partial W/\partial \varepsilon_{ij} \qquad (11)$$

$$\left.\begin{aligned} \Delta W &= \sigma_{ij}\Delta\varepsilon_{ij} + \Delta_2 W \\ \Delta_2 W &= \tfrac{1}{2}(\,\partial^2 W/\partial\varepsilon_{ij}\partial\varepsilon_{ml})\,\Delta\varepsilon_{ij}\,\Delta\varepsilon_{ml} \\ &\quad + \tfrac{1}{6}(\,\partial^3 W/\partial\varepsilon_{ij}\partial\varepsilon_{ml}\partial\varepsilon_{pq})\,\Delta\varepsilon_{ij}\,\Delta\varepsilon_{ml}\,\Delta\varepsilon_{pq} + \ldots \end{aligned}\right\} \qquad (12)$$

$$\left.\begin{aligned} \Delta\sigma_{ij} &= \partial\Delta_2 W/\partial\Delta\varepsilon_{ij} \\ \partial\Delta_2 W/\partial\varepsilon_{ij} &= \Delta\sigma_{ij} - (\,\partial^2 W/\partial\varepsilon_{ij}\partial\varepsilon_{ml})\,\Delta\varepsilon_{ml} \end{aligned}\right\} \qquad (13)$$

$$\begin{aligned} (\Delta W)_{,1} &= (\partial\Delta\, W/\partial\varepsilon_{ij})\varepsilon_{ij,1} + (\partial\Delta\, W/\partial\Delta\varepsilon_{ij})\Delta\varepsilon_{ij\,,1} \\ &= \Delta\sigma_{ij}\,\varepsilon_{ij,1} + (\,\sigma_{ij} + \Delta\,\sigma_{ij})\,\Delta\varepsilon_{ij,1} \end{aligned} \qquad (14)$$

$$\Delta(\,\sigma_{ij}\, u_{i,1})_{,j} = \Delta\sigma_{ij}\, u_{i,1j} + (\sigma_{ij} + \Delta\sigma_{ij})\,\Delta u_{i,1j} \qquad (15)$$

If there are no singular point, etc. in the region V enclosed by Γ then by using Gauss divergence theorem the eqn.(10) gives

$$\Delta J = \int_V \{\Delta W_{,1} - \Delta(\,\sigma_{ij}\, u_{i,1})_{,j}\}\, dv \qquad (16)$$

Using eqns. (14), (15) we easily know $\Delta J = 0$, i.e. ΔJ is also path independent. Eqn (10) may also written as

$$\Delta J = \int_V \{(\sigma_{ij}\,\Delta\varepsilon_{ij} + \Delta_2 W) n_1 - (T_i\,\Delta u_{i,1} + \Delta T_i u_{i,1} + \Delta T_i\,\Delta u_{i,1})\}\, ds \qquad (17)$$

$$\begin{aligned} &= \int_\Gamma \{\,(\sigma_{ij}\,\Delta\varepsilon_{ij} + \Delta_2 W)\, n_1 - (\,T_i + \Delta T_i)\,\Delta u_{i,1}\}\, ds\; - \\ &\quad - \int_V \{\,\sigma_{ij,1}\,\Delta\varepsilon_{ij} + (\partial\Delta_2 W/\partial\varepsilon_{ij})\,\varepsilon_{ij,1}\}\, dv \end{aligned} \qquad (18)$$

We note that in general case $\partial\Delta_2 W/\partial\varepsilon_{ij} \neq 0$. If the incremental load at every step is small then eqn.(17) is reduced to eqn.(19) or eqn.(20)

$$\Delta J = \int_\Gamma \Delta\Phi\, ds\ ; \quad \Delta\Phi = \sigma_{ij}\Delta\varepsilon_{ij}\, n_1 - T_i\Delta u_{i,1} - \Delta T_i u_{i,1} \qquad (19)$$

$$\dot J = \int_\Gamma \dot\Phi\, ds\ ; \quad \dot\Phi = \sigma_{ij}\,\dot\varepsilon_{ij}\, n_1 - T_i\,\dot u_{i,1} - \dot T_i u_{i,1} \qquad (20)$$

where $\dot J = dJ/d\lambda$, λ may be time or loading parameter, and

$$J = \int_o^\lambda \dot J\, d\lambda \qquad (21)$$

If ΔJ is path independent, then J is also path independent.

INCREMENTAL J-INTEGRAL IN ELASTO-PLASTICITY

Let eqns. (17)-(20) also be the definition of the incremental J-integral in the elasto-plastical theory. Fig.3 shows the deformation process from the state M to M + 1. In this process

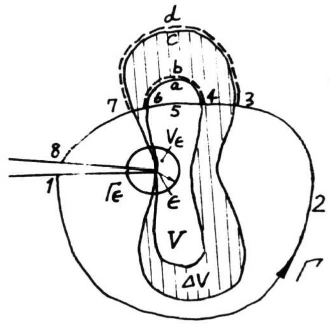

Fig.3

the values of variables on the connective boundary between the elastic and plastic regions may be discontinuous. But they must subjected to some restrictions. At the boundaries of the state M and M+1 we have

Equilibrium conditions

$$[T_i] = [\Delta T_i] = [\sigma_{nn}] = [\sigma_{ns}] = [\Delta\sigma_{nn}] = [\Delta\sigma_{ns}] = 0 \qquad (22)$$

Continuous conditions

$$[u_i] = [\Delta u_i] = [\Delta u_n] = [\Delta u_s] = [\Delta u_{n,s}] = [\Delta u_{s,s}] = 0 \quad (23)$$

and the following relations may easily be constructed (Fig.1)

$$\left.\begin{aligned} &[\sigma_{11}] = [\sigma_{ss}]\, n_2^2 \,,\ [\sigma_{22}] = [\sigma_{ss}]\, n_1^2 \,, \\ &[\sigma_{12}] = -[\sigma_{ss}]\, n_1 n_2 \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} &[\Delta u_{1,1}] = [\Delta u_{n,n}] n_1^2 - [\Delta u_{s,n}] n_1 n_2 \,, \\ &[\Delta u_{1,2}] = [\Delta u_{n,n}] n_1 n_2 - [\Delta u_{s,n}]\, n_2^2 \,, \\ &[\Delta u_{2,1}] = [\Delta u_{n,n}] n_1 n_2 + [\Delta u_{s,n}]\, n_1^2 \,, \\ &[\Delta u_{2,2}] = [\Delta u_{n,n}] n_2^2 + [\Delta u_{s,n}] n_1 n_2 \end{aligned}\right\} \quad (25)$$

At the initial boundary of M state the material on two sides just begin plastic deformation so we have

$$[\Delta\varepsilon_{ij}] = [\Delta u_{i,1}] = 0 \quad \text{(at most equal to const.}(\Delta\sigma_{ij})^2) \quad (26)$$

At the new boundary of M + 1 state we also have

$$[\sigma_{ij}] = [u_{i,1}] = 0 \quad \text{(at most equal to const.}\Delta\sigma_{ij}) \quad (27)$$

In eqns.(22)-(27) σ_{nn}, σ_{ns}, σ_{ss},... correspond to the normal, shear and tangential stress.... respectively at the boundary curve. $[F]=F^+ - F^-$, where F^+ and F^- are the values of F at the left and right sides of the counterclockwise path (boundary) respectively. Let the integral path is $\Gamma_{12345678}$ in eqn. (19) (Fig.3) then

$$\Delta J = \int_{\Gamma_{12345678}} = \int_{\Gamma_{123d78}} + \oint_{\Gamma_{34b67c3}} + \oint_{\Gamma_{456a4}} + \left(\int_{\Gamma_{4a6}} - \int_{\Gamma_{4b6}}\right) + \left(\int_{\Gamma_{3c7}} - \int_{\Gamma_{3d7}}\right) \quad (28)$$

Analyze the eqn. (28) term by term

(1) $$\int_{\Gamma_{3c7}} - \int_{\Gamma_{3d7}} = \int_{\Gamma_{3c7}} \{ [\sigma_{ij}\Delta\varepsilon_{ij}] n_1 - [T_i \Delta u_{i,1}] - [\Delta T_i\, u_{i,1}] \} ds \quad (29)$$

where Γ_{3c7} is part of the new boundary. Substituting eqns. (22)-(25) and (27) into eqn. (29) we know the term $[\Delta T_i\, u_{i,1}]$ may be neglected and

$$\int_{\Gamma_{3c7}} - \int_{\Gamma_{3d7}} = \int_{\Gamma_{3c7}} \{ \sigma_{ij}[\Delta\varepsilon_{ij}]\, n_1 - T_i[\Delta u_{i,1}] \} ds = 0 \quad (30)$$

Similarly we have

(2) $$\int_{\Gamma_{4a6}} - \int_{\Gamma_{4b6}} = \int_{\Gamma_{4a6}} \{ \Delta\varepsilon_{ij} [\sigma_{ij}] - \Delta T_i [u_{i,1}] \} ds$$

where Γ_{4a6} is part of the initial boundary of M state.

(3) $$\oint_{\Gamma_{34b67c3}} = -\int_{\Delta V} \{ \sigma_{ij,1}\Delta\varepsilon_{ij} - \varepsilon_{ij,1}\Delta\sigma_{ij} \} dv \quad (32)$$

where ΔV is the new incremental plastic region.

(4) $$\oint_{\Gamma_{456a4}} = -\int_{V_1} \{ \sigma_{ij,1}\Delta\varepsilon_{ij} - \varepsilon_{ij,1}\Delta\sigma_{ij} \} dv \quad (33)$$

where V_1 is part of the initial plastic region enclosed by Γ_{456a4}. For the elastic material we can easily prove that the eqns.(31)-(33) are all equal to zero. For the elasto-plastic material there are no strong discontinuous boundary of σ_{ij}, ε_{ij}... so we have the following results.

At the initial boundary the values of $[\sigma_{ij}]$, $[u_{i,1}]$ are all of the order $\Delta\sigma_{ij}$, so that the value of eqn.(31) is of the order $(\Delta\sigma_{ij})^2$. Because ΔV is proportional to $\Delta\sigma_{ij}$, the value of eqn.(32) is also of order $(\Delta\sigma_{ij})^2$. In the initial plastic region if deformation theory of plasticity can be applied then eqn.(33) is exactly equals to zero. For complex loading and unloading cases the value of eqn.(33) may be finite. From the above discussion in the general case we introduce a new parameter ΔJ_t or $\dot{J}_t$ characterizing the behavior of a crack end:

$$\Delta J_t = \int_{\Gamma_\varepsilon} \Delta\Phi\, ds = \Delta J - \int_{\Gamma_\rho} \Delta\Phi\, ds = \Delta J - \int_V \{ \sigma_{ij,1}\Delta\varepsilon_{ij} - \varepsilon_{ij,1}\Delta\sigma_{ij} \} dv \quad (34)$$

$$\dot{J}_t = \int_{\Gamma_\varepsilon} \dot{\Phi}\, ds = \dot{J} - \int_{\Gamma_\rho} \dot{\Phi}\, ds = \dot{J} - \int_V \{ \sigma_{ij,1}\dot{\varepsilon}_{ij} - \varepsilon_{ij,1}\dot{\sigma}_{ij} \} dv \quad (35)$$

where ΔJ and $\dot{J}$ are determined by (19) and (20) with a arbitrary path respectively. $V = \Sigma V_i$ is the sum of the regions V_1, V_2,.... where occur or had occured the plastic deformation located at the interior to Γ but exterior to Γ_ε. $\Gamma_\rho = \Sigma\Gamma_i$ is the boundaries of V and positive direction of Γ_i is selected such that V_i is always located at the left side alon Γ_i. Γ_ε is a path enclosing the tip with radius $\epsilon \to 0$ for ideal crack or a path departs from boundary a small distance $\epsilon \to 0$ for a blunt crack. From the physical view, ϵ is a small but finite value determined by the behavior of materials (Kuang,1982). Obviously ΔJ_t, $\dot{J}_t$ are path independent. If in V we have

$$\sigma_{ij,1}\Delta\varepsilon_{ij} - \varepsilon_{ij,1}\Delta\sigma_{ij} = 0 \quad (36)$$

then ΔJ and $\dot{J}$ are all path independent. The condition (36) is less restriction than the condition of proportional loading. In the finite element calculation the incremental load usualy is small but finite, so we need change eqns.(17), (18), (34)-(36) into the form of finite deformation and ΔJ_t is calculated by it.

For ideal plasticity, eqns.(29), (31) may take finite values, so that we need account it in calculating ΔJ.

For a general multi-connected domain the first two equality of equations (34), (35) are also valid. It is importent for problems of inclusions.

J-INTEGRAL IN THE FINITE DEFORMATION

In the finite element method for the incremental elasto-plastic theory the updated lagrange method is usually applied. we also apply this method to discuss the J-integral in the finite deformation. Let the initial state is C_o . We can define the ΔJ-integral at the N+1 state as

$$\Delta J_{N+1} = \int_{\Gamma} \{\Delta W^N n_1^N - \Delta (t_{ji}^N \, n_j^N \, u_{i,1}^N)\} \, ds^N \tag{37}$$

where ΔJ_{N+1} is produced by the Nth incremental loading. t_{ji} are the lagrange stress components and equal to the Euler stress components σ_{ji} at the state N but Δt_{ji}^N not equal to $\Delta \sigma_{ji}^N$.

$$\Delta W = t_{ji} \Delta u_{i,j} + \Delta_2 W \tag{38}$$

$$\Delta_2 W = (\Delta t_{ji} \, \Delta u_{i,j})/2 \tag{39}$$

The eqn.(37) is the natural extension of the eqn.(10). It is easy to prove that for nonlinear elasticity ΔJ_{N+1} is path independent. If the incremental load at every step is small then we have

$$\left.\begin{aligned} &\Delta J_{N+1} = \int_{\Gamma} \Delta \Phi_1 \, ds \; ; \\ &\Delta \Phi_1 = t_{ji} \, \Delta u_{i,j} n_1 - t_{ji} \, n_j \Delta u_{i,1} - \Delta t_{ji} \, n_j \, u_{i,1} \end{aligned}\right\} \tag{40}$$

In similar way to the last section, for elasto-plasticity we can introduce

$$\begin{aligned} \Delta J_t &= \int_{\Gamma_\epsilon} \Delta \Phi_1 \, ds = \Delta J - \int_{\Gamma_p} \Delta \Phi_1 \, ds \\ &= \int_{\Gamma_\epsilon} \{ n_1 (t_{ji} \Delta u_{i,j} + \tfrac{1}{2} \Delta t_{ji} \Delta u_{i,j}) - \\ &\quad - n_j (t_{ji} + \Delta t_{ji}) \Delta u_{i,1} - n_j \Delta t_{ji} \, u_{i,1} \} d\Gamma \qquad (41) \\ &= \int_{\Gamma} \{ n_1 \Delta W - n_j (t_{ji} + \Delta t_{ji}) \Delta u_{i,1} - n_j \Delta t_{ji} \, u_{i,1} \} d\Gamma \\ &\quad + \int_V \{ \Delta t_{ji} (\sigma_i \; u_{i,j1} + \tfrac{1}{2} \Delta u_{i,j1}) - \\ &\quad - \Delta u_{i,j} (t_{ji,1} + \tfrac{1}{2} \Delta t_{ji,1}) \} dv \qquad (42) \end{aligned}$$

For the small incremental loading we have

$$\Delta J_t = \Delta J - \int_V (t_{ji,1} \Delta u_{i,j} - u_{i,j1} \Delta t_{ji}) \, dv \tag{43}$$

If there exist some plastic regiogs $V = \Sigma V_i = V_1 + V_2 + \ldots + V_l$ and the corresponding boundaries are $\Gamma_p = \Sigma \Gamma_i = \Gamma_1 + \ldots + \Gamma_l$ then

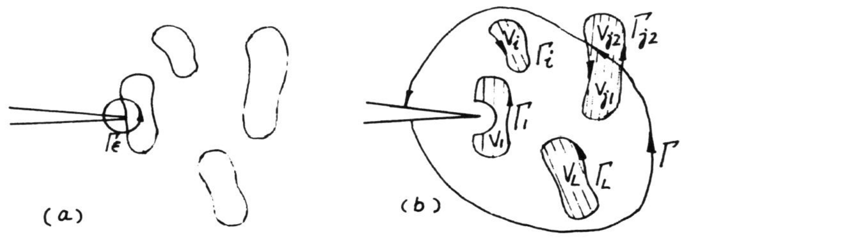

Fig. 4

$\Delta J_t = \int_{\Gamma_\epsilon} \Delta \Phi_1 ds$ (Fig.4a) is equal to $(\int_{\Gamma} - \int_{\Gamma_1} - \int_{\Gamma_i} - \int_{\Gamma_{j1}} - \ldots - \int_{\Gamma_L}) \Delta \Phi_1 ds$ (Fig.4b) or equal to $\int_{\Gamma} \Delta \Phi_1 ds - (\int_{V_1} + \int_{V_2} + \ldots + \int_{V_i} + \ldots + \int_{V_{j1}} + \ldots + \int_{V_L})(t_{ji,1} \Delta u_{i,j} - u_{i,j1} \Delta t_{ji}) dv$.

We also note that in the above equations n_j are the direction consines of a unit normal to Γ in the N state. Γ is composed of the same particles but occupies the different locations at the different state.

From the above discussion we can find that the J-integral in the region where are occured elastic and plastic deformation at different parts is rather similar to the contour integral of function holomorphic in multi- connected domain.

CONCLUSION

We proposed a new parameter characterized the crack tip behavior ΔJ_t and $J_t = \Sigma \Delta J_t$. J_t could be instead of the usual parameter J as a fracture criterion in the general elasto-plastic deformation. If $t_{ji,1} \Delta u_{i,j} - u_{i,j1} \Delta t_{ji} = 0$ everywhere then $J_t = J_{usual}$. else $J_t \neq J_{usual}$.

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