# SOME CRACK PROBLEMS ON PLATES AND SHELLS

**Liu Chuntu and Li Yingzhi**

*Institute of Mechanics, Academia Sinica, China*

## ABSTRACT

This paper presents a summary of the recent work of the authors in following areas: (1) The stress-strain fields at the crack tip in a Reissner's plate. (2) The stress-strain fields at the crack tip in a Reissner's shell. (3) The calculation of the stress intensity factors for finite size plates.

## INTRODUCTION

The study of bending cracked plates and shells is one of the fundamental problems in engineering. The problem is of considerable importance in many areas, such as aerospace industry, chemical industry etc. In the earlier literature the classical theory was used (Williams, 1961; Folias, 1965). In recent years many investigators began to study the problem with Reissner's theory (Reissner, 1947). Knowles and Wang(1960), Hartranft and Sih(1968) indicated the singularity of Reissner's plate. Murthy (1981) found the expansions of stress strain fields at crack tip for symmetric case. Yu and Yang (1982) given an asymptotic solution of zero order. For a finite size plate in bending, the stress intensity factors for Mode I were clculated(Barsoum, 1976; Rhee and Atluri, 1981; Li Yingzhi and Liu Chuntu, 1981). The stress intensity factors in infinite plate with uniform twisting moment were calculated using integral transformation (Wang,1968; Delate and Erdogan, 1979). The stress intensity factors for mixed mode in a finite plate using Reissner's theory were obtained by Liu Chuntu and Li Yingzhi (1983).

Since curvature exists in shells, extension and bending are coupled, which makes the problem very difficult. Recently, the Reissner's shell theory was used and a ten-order differential equation was derived. Since the problem is complicated only the first term of expansion was given(Sih and Hagendorf, 1973). In order to calculate stress intensity factors (especially for mixed mode), the expansion of the stress-strain fields at the crack ti was proposed (Liu Chuntu and Li Yingzhi, 1984).

In recent years, the authors studied the stress-strain fields in Reissner's plates and shells using the eigenfunction expansion method. Based on the results obtained, the high order special elements are used to calculate the stress intensity factors for plates and shells. In our experience, this is one of the effective methods for analysis of cracked plates and shells.

## THE STRESS FIELDS AT CRACK TIP AND STRESS INTENSITY FACTORS IN REISSNE'S PLATE

### The Stress Strain Fields at Crack Tip

A plate containing a semi-infinite crack in bending is shown in Fig. 1.

Based on Reissner's theory, the governing equations could be expressed in terms of three generalized displacements $\psi_x$, $\psi_y$ and W as follows (Hu, 1981)

$$D\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+C\left(\frac{\partial W}{\partial x}-\psi_x\right)=0 \qquad (1)$$

$$D\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)+C\left(\frac{\partial W}{\partial y}-\psi_y\right)=0 \qquad (2)$$

$$C\left(\frac{\partial^2 W}{\partial x^2}+\frac{\partial^2 W}{\partial y^2}-\frac{\partial\psi_x}{\partial x}-\frac{\partial\psi_y}{\partial y}\right)+P=0 \qquad (3)$$

Where $D=\frac{Eh^3}{12(1-\nu^2)}$ is bending stiffness

$C=\frac{5}{6}Gh$ is shearing stiffness

The boundary conditions are

When $\theta=\pm\pi$

$$M_\theta = M_{r\theta} = Q_\theta = 0 \qquad (4)$$

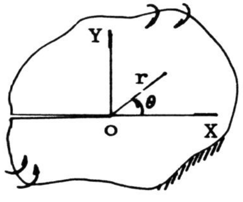

Fig. 1

In order to obtain the stress-strain fields at the crack tip. we could use the following two methods:

(1) The double eigenexpansion method (Liu Chuntu, 1981)

The generalized displacements $\psi_r$, $\psi_\theta$ and W could be expanded in double power series.

$$\psi_r^{(\lambda)} = r^\lambda\left(a_0^{(\lambda)}(\theta)+a_1^{(\lambda)}(\theta)r+a_2^{(\lambda)}(\theta)r^2+\ldots\ldots\right)$$

$$\psi_\theta^{(\lambda)} = r^\lambda\left(b_0^{(\lambda)}(\theta)+b_1^{(\lambda)}(\theta)r+b_2^{(\lambda)}(\theta)r^2+\ldots\ldots\right)$$

$$W^{(\lambda)} = r^\lambda\left(C_0^{(\lambda)}(\theta)+C_1^{(\lambda)}(\theta)r+C_2^{(\lambda)}(\theta)r^2+\ldots\ldots\right) \qquad (5)$$



Substituting eq. (5) into eq. (1)-(4), comparing with the terms of $O(r^{\lambda-2})$, we have

$$(\lambda^2-1)a_0+\frac{1-\nu}{2}a_0''+\left(\frac{1+\nu}{2}\lambda-\frac{3-\nu}{2}\right)b_0'=0$$

$$\left(\frac{1+\nu}{2}\lambda+\frac{3-\nu}{2}\right)a_0'+\frac{1-\nu}{2}(\lambda^2-1)b_0+b_0''=0$$

$$C_0''+\lambda^2 C_0=0 \qquad (6)$$

The corresponding boundary conditions are:

When $\theta=\pm\pi$

$$(1+\nu\lambda)a_0+b_0'=0$$

$$a_0'+(\lambda-1)b_0=0$$

$$C_0'=0 \qquad (7)$$

From eq.(6), the solutions of $a_0(\theta)$, $b_0(\theta)$, $c_0(\theta)$ could be found Substituting these solutions into eq. (7), the linear equations whose unknowns are the coefficients of the expansions could be obtained. In order to satisfy these equations, we let

$$\lambda=\frac{n}{2} \qquad n=0,1,2,\ldots. \qquad (8)$$

Comparing with the terms of $O(r^{\lambda-1})$, $O(r^{\lambda})$,... respectively, the asymptotic governing equations and bounary conditions could be found. These governing equations are inhomogeneous. The solutions of the corresponding homogeneous equations have some regularities.

With the solution of $a_i(\theta)$, $b_i(\theta)$, $c_i(\theta)$ known, the expansion of $\psi_r$, $\psi_\theta$ and W could be obtained. Based on the asymptotic solution of the crack tip displacement fields, a high-order special element was proposed to obtain the bending stress intensity factor for a finite plate (Li Yingzhi and Liu Chuntu, 1981).

Since the problems is complicated, only the first several terms of the expansion were given (Liu Chuntu, 1981). In order to obtain the general solution, a better method was proposed.

(2) Displacement Function Method (Liu Chuntu and Li Yingzhi, 1983)

Hu (1981) introduced two displacement functions F and f:

$$\psi_x=\frac{\partial F}{\partial x}+\frac{\partial f}{\partial y} \qquad \psi_y=\frac{\partial F}{\partial y}-\frac{\partial f}{\partial x} \qquad (9)$$

Substituting eq. (9) into eq. (1)-(2), we have

$$\frac{\partial}{\partial x}\left[D\nabla^2F+C(W-F)\right]+\frac{\partial}{\partial y}\left[D\frac{1-\nu}{2}\nabla^2 f-cf\right]=0 \qquad (10)$$

$$\frac{\partial}{\partial y}\left[D\nabla^2F+C(W-F)\right]-\frac{\partial}{\partial x}\left[D\frac{1-\nu}{2}\nabla^2 f-cf\right]=0 \qquad (11)$$

This is Cauchy-Riemann equation, from which it follows that:

$$\frac{1-\nu}{2} D\nabla^2 f - cf + i\left[D\nabla^2 F + c(w-F)\right] = c\,\Phi(x+iy) \quad (12)$$

Where $\Phi$ (x+iy) is an analytic function. Hu(1981) assumed that $\Phi$ (x+iy)=0, it is correct for cases without singularity. As the crack tip is a singular point, general speaking, $\Phi$ (x+iy) $\neq$ 0.

Separating real part and imaginary part in eq.(12), we have

$$\nabla^2 f - 4k^2 f = 4k^2 \, Re\,\Phi \quad (13)$$

$$w = F - \frac{D}{c}\nabla^2 F + Im\,\Phi \quad (14)$$

Where $$4k^2 = \frac{2c}{D(1-\nu)}$$

Substituting eq.(9), (14) into eq.(3), we have

$$D\nabla^2\nabla^2 F = P \quad (15)$$

For a cracked plate, the bending fracture problems are reduced to solving two equations (13), (15) in terms of F and f with the boundary conditions.

According to the singularity analysis (Knowles and Wang, 1960; Hartanft and Sih, 1968), the singularity of Mx, My, Mxy, $Q$ x, $Q$ y should be of O($r^{-\frac{1}{2}}$). These conditions demand that F and f should be of O($r^{\frac{3}{2}}$) and W should be of O($r^{\frac{1}{2}}$). These are the singularity conditions at crack tip. Only by letting $\Phi$ (x+iy) $\neq$ 0 could we find a solution which satisfies all the singularity conditions for mixed mode.

The function $\Phi$ (x+iy) could be expanded in series

$$\Phi(x+iy) = \sum_{\mu} (\beta_\mu + i\alpha_\mu) z^{\mu} = \sum_{\mu} (\beta_\mu + i\alpha_\mu) r^{\mu} (\cos\mu\theta + i\sin\mu\theta) \quad (16)$$

The solution of eq. (13), (15) could be expressed in the sum of a particular solution and the general solution of the corresponding homogeneous equations.

The particular solution could be chosen as follows

$$f_1 = -Re\,\Phi \qquad F_1 = 0 \quad (17)$$

The homogeneous equation corresponding to eq. (13) is

$$\nabla^2 f_o - 4k^2 f_o = 0 \quad (18)$$

When P = 0 , from eq. (15), we have

$$D\nabla^2\nabla^2 F = 0 \quad (19)$$

Eq. (19) is a biharmonic equation, let

$$F(r,\theta) = \sum_{\lambda} r^{\lambda+1} F(\theta) \quad (20)$$



Eq. (18) is a Helmholtz equation, function $f_o$ could be expressed in modified Bessel functions. From the condition of finite strai energy, we should drop out the modified Bessel functions of second kind and $f_o$ could be expressed in modified Bessel functions of first kind $I_\lambda(2kr)$ only.

For symmetric case

$$f_\lambda = \sin\lambda\theta \cdot I_\lambda(2kr) \quad (21)$$

For anti-symmetric case

$$f_\lambda^* = \cos\lambda\theta \cdot I_\lambda(2kr) \quad (22)$$

The linear combination of $f_\lambda$ , $f_\lambda^*$ is also a solution of eq. (18 For convenience the general solution of eq. (18) is expressed in the following linear combination

$$f_o = \sum_{\lambda}\sum_{n=0,1,\cdots} (A_{\lambda-1+2n} f_{\lambda-1+2n} + B_{\lambda-1+2n} f^*_{\lambda-1+2n}) \quad (23)$$

Substituting eq. (20), (23) into eq. (18), (19), the linear equations whose unknowns are the coefficients of the expansions coul be obtained. In order to satisfy these equations, we let

$$\lambda = \pm\frac{n}{2} \qquad n = 0,1,2,\ldots \quad (24)$$

With the condition of finite strain energy, $\lambda$ should be positive only. By using the boundary conditions, the relations between coefficients in eigenfunction expansion could be found. With the expression of F and f known, the expressions of $\psi_r$ , $\psi_\theta$ and W as well as Mr, $M_\theta$ , $M_{r\theta}$ ,$Q_r$ ,$Q_\theta$ could be obtained.

## Numerical Examples

Example 1. Infinite plate with uniform bending moment

This problem was studied by Hartranft and Sih (1968). The stres intensity factor is

$$K_{1(z)} = \frac{12z}{h^3}\Phi_{(1)} M\sqrt{\pi a} \quad (25)$$

The maxium value takes place at z = h/2.

$$K_I = \frac{6M}{h^2}\Phi_{(1)}\sqrt{\pi a} \quad (25')$$

In order to simulate the infinite plate, the plate's semilength L should be larger than 20a. The graph and results are shown in Fig.2 and Fig.3 respectively.

Example 2 Finite plate with uniform bending moment (Li Yingzhi and Liu Chuntu, 1981)

In order to investigate the variation of sterss intensity factor of finite plate with different thickness and width, the stress intensity factors for a/L = 0.1, 0.2, 0.4 and 0.5 are calculated The uniform bending moment is taken as 1 kg-cm/cm, crack semi-

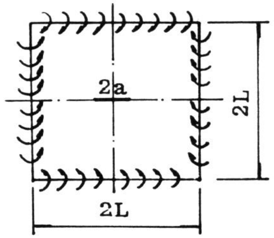


Fig.2

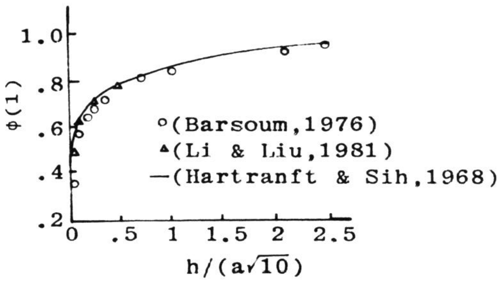


Fig.3

length a = 1 cm. The results are shwon in Fig.4 and Fig.5 respectively.

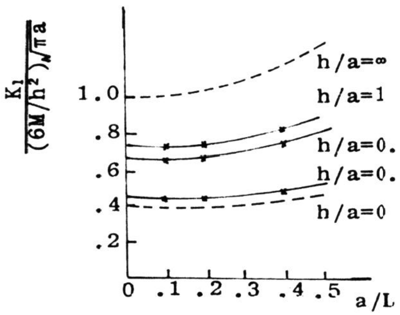


Fig.4

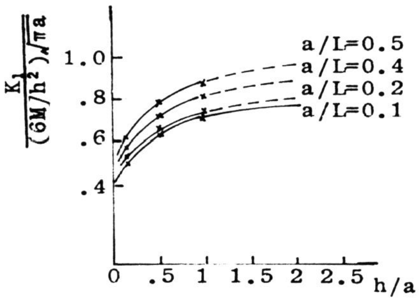


Fig.5

Example 3. The effect of boundary conditions on the stress intensity factors (Li Yingzhi and Liu Chuntu, 1981)

In order to compare with the effect of different boundary conditions on the stress intensity factors, the calculations of the simple supported plate and free plate are carried out. In the calculation the bending moment is taken as 1 kg-cm/cm and h/a=1. The results are shown in Fig.6

Example 4. Finite plate with uniform twisting moment (Liu Chuntu and Li Yingzhi, 1983)

The infinite plate with uniform twisting moment was studied by Wang (1968), Delate and Erdogan (1979). For finite plate, the numberical graph and results are shown in Fig.7 and Fig.8. In Fig.8 the solutions for a/L = o is obtained by extrapolating method, which represents the solution for infinite plate and



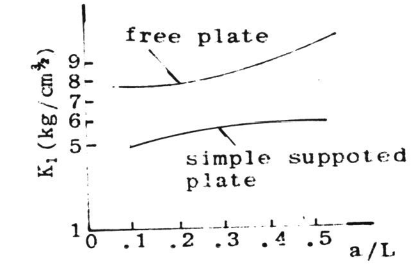


Fig. 6

compares favourably with that results (Delate and Erdogan, 1979).

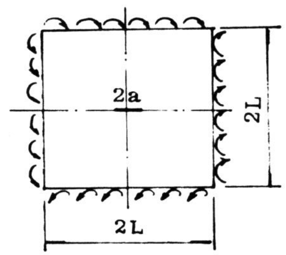


Fig. 7

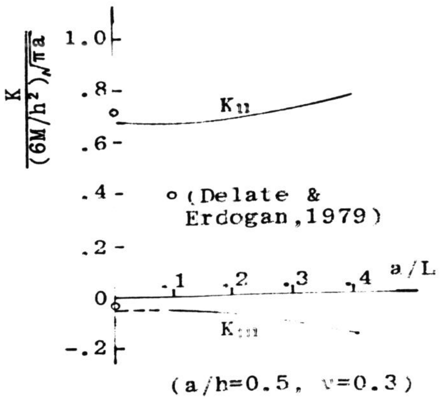


Fig. 8

## THE STRESS STRAIN FIELDS AT CRACK TIP IN CRACKED SPHERICAL SHELL

### The Governing Equation of a Cracked Spherical Shell And Their Simplified Forms

A spherical shell containing a through crack is shown in Fig. 9 with the crack tip at the origin of the coordinates. The shallo shell theory, taking into account of shear deformation, could be expressed as follows (Hu, 1981).

The governing equations are

$$D\left(\frac{\partial^2\psi_x}{\partial x^2}+\frac{1-\nu}{2}\frac{\partial^2\psi_x}{\partial y^2}+\frac{1+\nu}{2}\frac{\partial^2\psi_y}{\partial x\partial y}\right)+C\left(\frac{\partial w}{\partial x}-\psi_x\right)=0 \qquad (26)$$

$$D\left(\frac{1+\nu}{2}\frac{\partial^2\psi_x}{\partial x\partial y}+\frac{1-\nu}{2}\frac{\partial^2\psi_y}{\partial x^2}+\frac{\partial^2\psi_y}{\partial y^2}\right)+C\left(\frac{\partial w}{\partial y}-\psi_y\right)=0 \qquad (27)$$

$$C\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}-\frac{\partial \psi_x}{\partial x}-\frac{\partial \psi_y}{\partial y}\right)+K\nabla^2\varphi+q=0 \tag{28}$$

Where k is the curvature, $\varphi$ is the stress function and

$$N_x=\frac{\partial^2\varphi}{\partial y^2} \qquad N_y=\frac{\partial^2\varphi}{\partial x^2} \qquad N_{xy}=-\frac{\partial^2\varphi}{\partial x\partial y}$$

The compatibility equation is

$$\frac{1}{B}\nabla^2\nabla^2\varphi + K\nabla^2 W = 0 \tag{29}$$

Where B is the in-plate stiffness.

Introducing displacement functions F and f, let

$$\psi_x=\frac{\partial F}{\partial x}+\frac{\partial f}{\partial y} \qquad \psi_y=\frac{\partial F}{\partial y}-\frac{\partial f}{\partial x} \tag{30}$$

Substituting eq. (30) into eq. (26), (27), we have

$$\frac{\partial}{\partial x}\left[D\nabla^2F+C(W-F)\right]+\frac{\partial}{\partial y}\left[\frac{D}{2}(1-\nu)\nabla^2 f-Cf\right]=0 \tag{31}$$

$$\frac{\partial}{\partial y}\left[D\nabla^2F+C(W-F)\right]-\frac{\partial}{\partial x}\left[\frac{D}{2}(1-\nu)\nabla^2 f-Cf\right]=0$$

Eq. (31) is Cauchy-Riemann equation from which it follows that

$$\frac{D}{2}(1-\nu)\nabla^2 f-Cf+i\left[D\nabla^2F+C(W-F)\right]=C\Phi(x+iy) \tag{32}$$

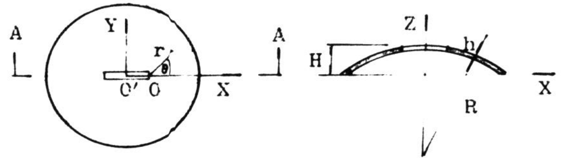


Fig. 9 A-A

Separating real part and imaginary part in eq. (32), we have

$$D\nabla^2F+C(W-F)=C\,\mathrm{Im}\,\Phi \tag{33}$$

$$\frac{D}{2}(1-\nu)\nabla^2 f - Cf = C\,\mathrm{Re}\,\Phi \tag{34}$$



From eq. (33), we have

$$W=F-\frac{D}{C}\nabla^2F+\mathrm{Im}\,\Phi \tag{35}$$

Substituting eq. (30), (35) into eq. (28), we have

$$D\nabla^2\nabla^2F-K\nabla^2\varphi=q \tag{36}$$

Substituting eq. (35) into eq. (29), we have

$$\frac{1}{B}\nabla^2\nabla^2\varphi+K\nabla^2F-K\frac{D}{C}\nabla^2\nabla^2F=0 \tag{37}$$

The governing equations could be reduced to three equations (34), (36) and (37) in terms of F, f and $\varphi$ . The function f, which is similar to that the bending plate case, is uncoupled. The functions F and $\varphi$ should satisfy two four-order differential equations.

If q = 0, from eq. (36), (37), we have

$$\nabla^2\nabla^2\nabla^2F-\frac{K^2B}{C}\nabla^2\nabla^2F+\frac{K^2B}{D}\nabla^2F=0 \tag{38}$$

In may be proved that, function F in eq. (38) is the sum of three functions $F_0$ , $F_1$ and $F_2$ , which should satisfy the following equations respectively.

$$F=F_0+F_1+F_2 \tag{39}$$

$$\nabla^2F_0=0 \tag{40}$$

$$\nabla^2F_1-4\lambda_1^2F_1=0 \tag{41}$$

$$\nabla^2F_2-4\lambda_2^2F_2=0 \tag{42}$$

Where

$$4\lambda_1^2=\frac{K^2B}{2C}+\sqrt{\frac{K^4B^2}{4C^2}-\frac{K^2B}{D}}\,,\qquad 4\lambda_2^2=\frac{K^2B}{2C}-\sqrt{\frac{K^4B^2}{4C^2}-\frac{K^2B}{D}}$$

With the F known, from eq. (36) the $\varphi$ could be obtained.

$$\varphi=\varphi_0+\frac{4D}{K}\left(\lambda_1^2F_1+\lambda_2^2F_2\right) \tag{43}$$

Where $\varphi_0$ is harmonic function, which should satisfy $\nabla^2\varphi_0=0$

From eq. (34), function f could be found as

$$f=f_0-\mathrm{Re}\,\Phi \tag{44}$$

$f_0$ should satisfy the following equation

$$\nabla^2 f_0 - 4\mu^2 f_0 = 0 \tag{45}$$

Where $4\mu^2 = \dfrac{2C}{D(1-\nu)}$

The boundary conditions are

When $\theta = \pm\pi$

$$M_\theta = M_{r\theta} = Q_\theta = N_{r\theta} = N_\theta = 0 \tag{46}$$

## The Eigenfunction Expansion of the Displacement Functions and Stress Function

The analytic function $\Phi$ could be expanded in series

$$\Phi(x+iy) = \sum_\mu (\beta_\mu + i\alpha_\mu) z^\mu = \sum_\mu (\beta_\mu + i\alpha_\mu) r^\mu (\cos\mu\theta + i\sin\mu\theta) \tag{47}$$

Harmonic function F could also be expanded in series

$$F_0 = \sum_\lambda r^{\lambda+1}\left[K^{(0)}_{\lambda+1}\cos(\lambda+1)\theta + L^{(0)}_{\lambda+1}\sin(\lambda+1)\theta\right] \tag{48}$$

Functions $f_0$ , $F_1$ and $F_2$ should satisfy the eq. (45), (41),(42) respectively. These equations are Helmholtz's equations. Their solutions could be expressed in modified Bessel functions. With the condition of finite energy, we must drop out the modified Bessel functions of second kind. The functions $f_0$ , $F_1$ and $F_2$ could then be expressed in modified Bessel functions of first kind only.

Similar to the bending cracked plate problem, substituting the expansion f, F and $\varphi$ into the boundary conditions, the linear equations whose unknowns are the coefficients of the expansions could be established. From these equations, the relations between the coefficients in the eigenfunction expansion could be found. With the functions f, F and $\varphi$ known, the generalized displacements and stesses could be obtained.

## REMARKS

1. Similar to the Williams' expansion in plane fracture problem, the general solutions of the stress-strain fields including Mode I, Mode II and Mode III at the crack tip for Reissner's plate and shell are given.

2. The general solutions for stress-strain fields at the crack tip in plates and shells provide a better mechanical foundation for calculation of stress intensity factors. The analytical methods for plane fracture problem could be adapted for the analyses of cracked plates and shells.



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