INFLUENCE OF COUPLE STRESSES ON STRESS INTENSITY FACTORS

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ABSTRACT

In elastic solids capable of sustaining couple stresses stress intensity factors (SIFs) depend on three additional parameters ν, ℓ/a and N. ν is Poisson's ratio, ℓ is a characteristic length of the material, a is crack length and N is a number representing the interaction of the microstructure with the displacement field. After a discussion of the significance of the limit $\ell/a \to 0$, a specific crack problem is solved and an 'effective' SIF (i.e., a closed-form expression for the SIF in the limit $\ell/a \to 0$) is obtained.

KEYWORDS

Couple stress; thermal stress; micropolar elasticity; Griffith crack; line crack; stress intensity factor; effective stress intensity factor.

INTRODUCTION

Fatigue tests (Peterson, 1953) show that the weakening effect of stress concentrations is diminished as the specimen size is reduced. In alternating bending fatigue tests, smaller the specimen size, the higher is the fatigue limit. Thus, reduction of specimen size seems to produce a 'stiffening effect'. Mindlin (1968) associated this effect with the severe strain gradients that might occur across the specimen due to the reduced size. He (Mindlin, 1962, 1968) extended the classical elasticity theory by incorporating the gradients of local rotation $w_{j,k}$ (where $\underline{w}$ = 1/2 curl $\underline{u}$; $\underline{u}$ = displacement) in the strain energy density function. In this theory the tensor $w_{j,k}$ is found to be associated with a couple stress tensor m_{ik} (couple per unit area) just as the strain tensor e_{ik} is associated with the usual stress tensor t_{ik}. The proportionality between $w_{j,k}$ and m_{ik} gives rise to additional elastic modulii. The ratio (say ℓ^2) of one of these to the shear modulus has the dimensions of length squared. Owing to the length scale ℓ the theory is capable of encompassing the 'size effects' mentioned above. This extended theory is called couple stress theory. Similar and other generalisations of the classical theory, incorporating couple stresses and hyper-stresses of increasing complexity and elusive physical reality, are available (Kroner, 1968; Sridharan, 1980 Chap 1 p.2); their origins go back to Voight and E. and F. Cosserat. Subsequently, in this paper, we shall be concerned with another extension of the classical theory - the well-known micropolar theory (Eringen, 1976) - which includes the couple stress theory as a particular 'extreme case'.

The stress concentration factor S_c for a circular hole (radius a) in a transverse tension field, worked out on the basis of couple stress theory (Mindlin and Tierstein, 1962), is found to depend on ℓ/a and Poisson's ratio (in classical elasticity S_c has the constant value 3). For $\ell/a > 0$, $S_c < 3$. As $\ell/a \to 0$, $S_c \to 3$. As ℓ/a increases S_c decreases, and for vanishingly small holes, the reduction in S_c is as much as 30 to 40 per cent depending on Poisson's ratio.

Impressed by the enormous reductions in S_c that the couple stresses can bring about, Sternberg and Muki (1967) investigated the singular stress concentrations at the tips of a Griffith crack (length 2a) situated in a transverse tension field. They found out that (i) the ordinary and couple stresses both have the $1/\sqrt{r}$ singularity; (ii) the stress environment at the crack-tips is controlled by two stress intensity factors, one for the ordinary stresses and the other for the couple stresses - $K^{(1)}$, $K^{(2)}$ respectively; (iii) $K^{(1)}$, $K^{(2)}$ are functions of, not only the load and crack length, but also ℓ/a and Poisson's ratio; and (iv) unlike S_c, the factor $K^{(1)}$ remains higher than its classical value when $\ell/a > 0$, and increases as ℓ/a decreases; in the limit $\ell/a \to 0$, $K^{(1)}$ shoots upto values 20 to 32 per cent more than the classical value. $K^{(2)} > 0$ for $\ell/a > 0$; and vanishes as $\ell/a \to 0$ (Ejike, 1969; Paul and Sridharan, 1980 a).

A SIGNIFICANT LIMIT

Notice that $K^{(1)}$ does not attain its classical value as $\ell/a \to 0$. This is not surprising to us - recall that the strain energy does attain its classical value (Atkinson and Lappington, 1977) - in view of the fact that in this limit a highest derivative term in a governing equation vanishes signifying the emergence of boundary layer effects. Therefore the limit $\ell/a \to 0$ does not represent a transition from couple stress theory to classical theory. The limiting values of $K^{(1)}$ are not applicable to continua which are incapable of sustaining couple stresses (as the classical one is) but applicable to those which can sustain couple stresses and yet have intrinsic length scale (ℓ) $\ll$ a. It is only the later type of continua that interest us, for, on physical grounds ℓ is believed to be of the order of grain size (Mindlin, 1968; Askar, 1972). Besides, in general, a continuum treatment is valid only if the intrinsic length scale of the medium (ℓ) is $\ll$ any physical linear dimension of the problem under consideration (here a). The limiting values of $K^{(1)}$, therefore, assume physical significance.

The result of Sternberg and Muki (1967) that the limits of $K^{(1)}$ are 20 to 32 per cent higher than the respective classical values, is clearly unrealistic. A method of solution more refined than the one used by these authors yields no better results; the said limits now range from 18 to 30 per cent (Atkinson and Lappington, 1977).

COUPLING NUMBER

Recently we (Paul and Sridharan, 1981) have considered the Griffith crack problem on the basis of micropolar theory of elasticity. This theory abandons the strain gradients, and introduces gradients of a vector field ϕ (called microrotation and assumed to be kinematically independent of $\underline{u}$) in the strain energy density function. In this theory, there appear a few intrinsic length scales - but only one, say ℓ, is relevant to the problem at hand - and a non-dimensional number N ($0 \leq N < \sqrt{2}$) which is a measure of the coupling of ϕ and the local rotation 1/2 curl $\underline{u}$. The number N, called the coupling number (Paul and Sridharan, 1980 b), has no analogue in couple stress theory. As $N \to 0$ classical theory is recovered; in the limit $N \to \sqrt{2}$ (the case of extreme coupling) couple stress theory can be deduced. This deduction brings to light certain inherent limitations of the couple stress theory.

THE LIMIT $\ell/a \to 0$ RECONSIDERED

The main result of Paul and Sridharan (1981) is that $K^{(1)}$ and $K^{(2)}$ depend on the coupling number N in addition to ℓ/a and Poisson's ratio. The limit $\ell/a \to 0$ now has two cases: (a) the simultaneous limit $\ell/a \to 0$, $N \to 0$ and (b) the limit $\ell/a \to 0$ with N arbitrary but fixed. We interpret limit (a) as a 'smooth' transition from couple stress material to classical material by a sequence of micropolar materials with decreasing micropolarity - micropolarity being determined by the parameters ℓ/a and N. In limit (a) the solution obtained by Paul and Sridharan (1981) - in particular the factor $K^{(1)}$ - attains the classical value. However the solution and the formula for $K^{(1)}$ are valid only for the case of 'weak micropolarity' ($\ell/a \ll 1$, $N \ll 1$), a condition more stringent than weak coupling (Sridharan, 1980). Limit (b) corresponds to $\ell/a \to 0$ of couple stress theory and has important physical significance as discussed earlier. Unfortunately limit (b) could not be applied to the solution - and to the factor $K^{(1)}$ - obtained by Paul and Sridharan (1981).

We now present a specific crack problem in which the crack-tip stress field is controlled by the force-stress intensity factor $K^{(1)}$ alone; in limit (b) this factor takes on an elegant closed form expression which may be called 'effective stress intensity factor'.

GRIFFITH CRACK INTERRUPTING HEAT-FLOW

Consider a Griffith crack, $-a \leq x_1 \leq a$, $x_3 = \pm 0$ (Cartesian coordinates x_i and plane strain in the x_1x_3 - plane) with thermally insulated faces, disturbing uniform heat-flux q_o in the negative x_3 - direction. In classical elasticity this problem was first considered by Florence and Goodier (1960) as a limiting case of the problem of heat-flow around an elliptic hole. See also (Sekine, 1977).

It is enough to consider the 'perturbed problem' : the half-plane ($x_3 \geq 0$) problem with the boundary conditions

$$t_{31}(x_1,0) = m_{32}(x_1,0) = 0\,, \qquad |x_1| \geq 0 \tag{1}$$

$$\begin{aligned} t_{31}(x_1,0) &= 0\,, \qquad |x_1| < a \\ u_1(x_1,0) &= 0\,, \qquad |x_1| > a \end{aligned} \tag{2}$$

$$\begin{aligned} T_{,3} &= -q_o\,, \qquad |x_1| < a \\ T &= 0\,, \qquad |x_1| > a \end{aligned} \tag{3}$$

(T = temperature) and the usual regularity conditions at ∞ ; for a unique solution and for bounded strain energy at the crack tips, an 'edge condition' should be imposed. We take this condition - in conformity with the classical solution - as

$$u_1(x_1,0) = O(\sqrt{a-x_1}), \quad x_1 \to a- \tag{4}$$

The field equations are

$$\begin{aligned} (\lambda+\mu)\,\text{grad div}\,\underline{u} + (\mu+\kappa)\nabla^2\underline{u} + \kappa\,\text{curl}\,\underline{\phi} - \beta_o\,\text{grad}\,T &= \underline{0} \\ (\alpha+\beta)\,\text{grad div}\,\underline{\phi} + \gamma\nabla^2\underline{\phi} + \kappa\,\text{curl}\,\underline{u} - 2\kappa\underline{\phi} &= \underline{0} \\ \nabla^2 T &= 0 \end{aligned} \tag{5}$$

Here λ, μ are Lamé constants; κ, α, β, γ are additional elastic modulii; $\beta_o = (3\lambda + 2\mu + \kappa)\alpha_t$ where α_t is the coefficient of linear thermal expansion. Let

$$\mu' = \mu + \tfrac{1}{2}\kappa\,, \quad N = \kappa/(\mu+\kappa), \quad \ell = \gamma/(2\mu+\kappa), \quad \nu' = \tfrac{1}{2}\lambda/(\lambda+\mu+\tfrac{1}{2}\kappa)$$

These quantities have the obvious physical meaning. We adopt the following notation for Fourier sine transform

$$\sqrt{\frac{2}{\pi}} \int_0^\infty f(v)\,\sin(vr)\,dv = F_s\,[f(v);\,r]$$

and a similar one for cosine transform. Further let

$$r = x_1/a, \quad y = \sqrt{v^2 + M^2}, \quad M = N/(\ell/a)$$

If we set

$$u_1(r,0) = a\,F_s\,[U(v);\,r]$$

the boundary conditions (1) and (2) yield - utilising the field eqns (5) and the stress-strain relations not quoted here - the dual integral equations

$$\begin{aligned} (a' + \tfrac{1}{2}N^2)\,F_s[v(1+G(v))\,U(v);\,r] &= (a'-1)\,F_s[U_1(v);\,r], \quad 0 \leq r < 1 \\ F_s[U(v);\,r] &= 0, \quad r > 1 \end{aligned} \tag{6}$$

in which

$$G(v) = \frac{\tfrac{1}{2}N^2}{a' + \tfrac{1}{2}N^2} \left\{ \frac{2v^2}{y(y+v)} - 1 \right\}$$

$$a' = [2(1-\nu')]^{-1}$$

The function $U_1(v)$ appearing in the right hand member of $(6)_1$

is to be determined from the thermal boundary conditions (3). We have

$$F_c[vU_1(v);\ r] = \sqrt{(\pi/2)}\ \beta_o\ a\ q_o\ (2\mu')^{-1}, \quad 0 \leq r < 1$$
$$F_c[U_1(v);\ r] = 0, \quad r > 1 \tag{7}$$

The solution is

$$U_1(v) = \sqrt{(\pi/2)}\ (\beta_o a q_o)\ (2\mu')^{-1}\ J_1(v)\ v^{-1} \tag{8}$$

where $J_1(.)$ is the Bessel function of order 1. Now, the only unknown is $U(v)$. Put

$$U(v) = \sqrt{\frac{2}{\pi}}\ \frac{a' - 1}{a' + \frac{1}{2}N^2}\ \frac{\beta_o\ a\ q_o}{2\mu'}\ \frac{\pi}{4}\ \int_0^1 \sqrt{t}\ \psi(t)\ J_1(tv)\ dt \tag{9}$$

This representation satisfies condition (4). The dual integral equations (6) yield a Fredholm integral equation:

$$\psi(t) + \int_0^1 K(u,t)\ \psi(u)\ du = t^{3/2}, \quad 0 \leq t \leq 1 \tag{10}$$

whose kernel $K(u,t)$, not quoted here, is symmetric and continuous. This formally completes the solution of the problem. The only non-vanishing stress on the plane of the crack is t_{31}. In the limit $r \to 1+$, we have

$$(2\mu')\ t_{31}(r,0) = (a'+(1/2)N^2)\ \sqrt{(2/\pi)}\ Q(r) + O(1)$$

where

$$Q(r) = \frac{d}{dr} \int_0^\infty [1+\cos(rv)]\ U(v)\ dv$$

Defining the Mode-II stress intensity factor as

$$K_{II} = 2a \lim_{r\to 1+} \sqrt{(r-1)}\ t_{31}(r,0)$$

we find

$$K_{II} = 1/2(1-a')\ \beta_O q_O a^{3/2}\ \psi(1) \tag{11}$$

As $N \to 0$, $\psi(1) \to 1$; and K_{II} attains its classical value say K_{II}^{o}. For arbitrary, but fixed N consider the limit $\ell/a \to 0$. In this limit

$$K(u,t) = \frac{-(1/2)N^2}{a' + \frac{1}{2}N^2}\ \sqrt{(ut)} \int_0^\infty x\ J_1(xu)\ J_1(xt)\ dx \tag{12}$$

provided the integral on the RHS is understood to define a generalised function. Let $H(.)$ denote the unit functional. We know

$$u \int_0^\infty J_1(xu)\ J_0(xt)\ dx = H(u-t)$$

Differentiating with respect to t

$$u \int_0^\infty x\ J_1(xu)\ J_1(xt)\ dx = \delta(u-t)$$

where $\delta(.)$ is the delta functional. Substituting in eqn(12)

$$K(u,t) = -\frac{(1/2)N^2}{a' + \frac{1}{2}N^2}\ \delta(u-t)$$

Eqn(10) is now readily solvable; the solution is

$$\psi(t) = [1+(1-\nu')\ N^2]t^{3/2}$$

Substituting in eqn(11) we obtain the 'effective stress intensity factor'

$$K_{II} = K_{II}^{O}\ [1+(1-\nu')N^2]$$

where K_{II}^{O} is the classical value of K_{II} and ν' is Poisson's ratio. The range of N is $(0,\sqrt{2})$.

For a Poisson's ratio of 0.33, and for N = .1, .2, .4, .6, .8, 1.0 the percentage increments in K_{II}, over the classical value are, respectively, 0.7, 2.7, 10.7, 24.1, 42.9 and 67.0.

For the case of extreme coupling ($N = \sqrt{2}$) the said increment is 150 per cent (Poisson's ratio = 0.25). The increment in K_{II} obtained on the basis of couple stress theory should coincide with this figure. But we are not in a position to check this as the solution to the persent problem on the basis of couple stress theory has not been obtained, as far as we know. However it is of interest to note that the increment in K_I for a penny-shaped crack worked out on the basis of couple stress theory, is 150 per cent (Ejike, 1969).

REFERENCES

Askar, A. (1972). Int. J Engng Sci. 10, 293-300.
Atkinson, C. and F.G. Leppington (1977). Int. J Solids Structures 13, 1103-1122.
Eringen, A.C. (Ed.) (1976). Continuum Physics IV. Academic Press, New York.
Ejike, U.B.C.O. (1969). Int. J Engng Sci. 7, 947-961.
Florence, A.L. and J. Goodier (1960). J App.Mech.,Trans. ASME. 635-639.
Kroner. E. (Ed.) (1968). Mechanics of Generalised Continuua, IUTAM symp. 1967. Springer-Verlag, Berlin.
Mindlin, R.D. and H.F. Tierstein (1962). Arch. Ration.Mech. Analysis 11, 415-448.
Mindlin, R.D. (1968). Influence of Couple-stresses on Stress Concentrations. In B.E. Rossi (Ed.), The William M. Murray Lectures 1953-1967, Soci. Exp. Stress Analysis.
Paul, H.S. and K. Sridharan (1980a). Int. J Engng Sci. 18, 651-664.
Paul, H.S. and K. Sridharan (1980b). Int. J Engng Sci. 18, 1431-1448.
Paul, H.S. and K. Sridharan (1981). Int. J Engng Sci. 19, 563-579.
Peterson, R.E. (1953). Stress Concentration Design Factors, Wiley, New York.
Sekine, H. (1977). Engng Fracture Mech. 2, 499-507.
Sridharan, K. (1980). Some Problems in Micropolar and Classical Elasticity - Ph.D. Thesis, Indian Inst. Tech., Madras, India.
Sternberg, E. and R. Muki (1967). Int. J Solids Structures 3, 69-95.