FINITE ELEMENT EVALUATION OF FRACTURE MECHANICS PARAMETERS USING RAPID MESH REFINEMENT

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ABSTRACT

This paper presents a simple and effective technique for the evaluation of fracture mechanics parameters. Finite elements are applied as a reliable tool for singularity problem solution. Because of the limitations on application on material and geometrical nonlinearity special elements are avoided. It is the rapid mesh refinement technique that enables accurate modelling of a cracked body response on the applied load. All classical elements are admissible. Simplex isoparameter triangular elements are applied here due to simplicity and universality in their formulation. A simple procedure, based on displacement field, is given for the evaluation of stress intensity factor and J-integral. The procedure requires linear variation of displacements inside an element. An analysis of the errors appearing during the solution process is given, enabling the optimized choice of the error influencing parameters. A final result is a possibility to use simple, commercially available programme and simplex elements for an accurate evaluation of fracture mechanics parameters with relatively small number of degrees of freedom and without limitation on material nature.

KEYWORDS

Finite elements, Rapid mesh refinement, $C^0$ convergency, Displacement formulation, Discretization and computational errors, Linear and nonlinear elastic behaviour, Elasto-plastic behaviour.

INTRODUCTION

Fracture mechanics parameters can be evaluated if the displacement or stress field in a certain domain containing a crack is known. The calculation of the displacement and stress field around crack is well-known problem in the theory of elasticity. The singularity problems are described through partial differential equations of $2m$th order and appropriate boundary conditions (Whiteman and Akin, 1979). The attention is restricted here on the second order ($m=1$) equations and prescribed displacements as a boundary condition. Therefore, the displacement formulation of finite elements is used and $C^0$ convergency requirements are essential.
It was proved (Fried and Yang, 1972) that the full rate of convergence is regained in this way. Consequently, the discretization error is $O(h^n)$ for $m=1$, $p=1$ and $m=1/2$ (linear elastic fracture mechanics).

The computational error mainly appears as a consequence of the round-off process in the computer memory. The effect of numerical integration and curved boundaries does not exist here because numerical integration is exact and there are no curved boundaries in the chosen example (for details of this effect see Clarrer and Raviart, 1972). Therefore, we concentrate here on the round-off error, given as (Fried, 1971b):

$$E_{\text{round-off}} = c_1 \cdot 10^{-5} \cdot C_s(K)$$

where $s$ is a number of decimals in the computer word, $C_s(K)$ is a spectral condition number of the global stiffness matrix and $c_1$ is a numerical constant. Using the relation for spectral condition number in the case of a nonuniform mesh (Fried, 1972), the round-off error can be expressed as:

$$E_{\text{round-off}} = c_1 \cdot 10^{-5} \cdot \max \left( h_{\text{max}}^N \right)$$

where $N$ is a number of elements, $h_{\text{max}}$ and $h_{\text{min}}$ are the maximum and minimum values for the diameters, respectively, and $c_1$ is a numerical constant.

It is useful to discuss now how the influence of the parameters on the magnitude of the errors. It is clear that the number of elements reduces the discretization error, the reduction being larger for the higher order elements (the larger $p$). However, the larger $p$ causes a larger number of elements, undesirable from the computational error point of view. It is impossible to find the "universally" best combination of the parameters influencing the magnitude of errors ($0$, $N$, $h_{\text{max}}/h_{\text{min}}$, $s$). The criteria for satisfying choice is (Fried, 1972):

$$10^{-5} \cdot C_s(K) < 1$$

**EVALUATION OF STRESS INTENSITY FACTORS AND J-INTEGRAL**

Knowing the displacement or the stress field in a domain containing a crack there are few possibilities for the stress intensity factor and J-integral evaluation. It is obvious how to choose between the displacement or the stress field. Since the finite element formulation here is based on the displacements, it is more accurate to use the displacement field for any further calculation. A standard technique for the stress intensity factor evaluation is the extrapolation of $K_I=K_I(r)$ curve to obtain the value for $K_I$ at $r=0$. This curve is obtained from the nodes along the radial direction $r$, using the relation for the plane problem:

$$K_I = \frac{Gv}{k} \left( \frac{r^2}{2} \right)$$

where $G$ is the shear modulus, $v$ is a displacement in y direction and $k$ is a parameter characterizing the type of a problem: $k=3-4v$ for the plane strain and $k=(3-\nu)/(1+\nu)$ for the plane stress, $\nu$ being Poisson's Ratio.
The starting point for J-integral evaluation is its definition as a path independent integral on the contour Γ:

\[ J = \int_{\Gamma} (\mathbf{W} \cdot \mathbf{n}) \, ds \]  

(6)

where \( \mathbf{W} \) is the strain energy given by \( W_{ij} = \frac{1}{2} \sigma_{ij} \epsilon_{ij} \), \( \sigma_{ij} \) and \( \epsilon_{ij} \) are components of the stress and strain tensors, \( \mathbf{n} \) is an outward normal on \( \Gamma \) and \( \frac{\partial \mathbf{u}}{\partial x} \) is a gradient of displacements.

It is essential to transform the relation (6) into the form dependent on the displacement field only. Such a transformation involves constitutive relations which are taken here as for a homogeneous linear isotropic body, as well as strain-displacement relations, which is taken here as for the small displacement gradients. For the plane problem the final relation for J-integral is the function of displacement gradients only (the relation which is not restricted on the plane problem can be find in Sedmak and co-workers, 1981):

\[ J = \int_{\Gamma} \left[ k_1 \left( \frac{\partial \mathbf{u}}{\partial x} \right)^2 + k_2 \left( \frac{\partial \mathbf{u}}{\partial y} \right)^2 \right] \, ds \]  

(7)

where \( k_1 \) and \( k_2 \) are constants characterizing the type of the problem:

- for the plane strain
  \[ k_1 = \frac{1}{1-z_0}, \quad k_2 = \frac{1}{z_0} \]  

(8)

- for the plane stress
  \[ k_1 = \frac{2}{1-v}, \quad k_2 = \frac{1-v}{v} \]  

(9)

The displacement gradients can be calculated directly from the known displacement field. Under the assumption of a linear variation of displacements, what is satisfied in the simplex finite elements, it is possible to transform the integral relation (7) into a sum suitable for the further calculation:

\[ J = \frac{1}{2} \sum_{j=1}^{N-1} \left( \mathbf{F} \cdot \mathbf{V}_{K,j} + \mathbf{F} \cdot \mathbf{X}_{K,j} \right) \]  

(10)

where \( \mathbf{F} \) and \( \mathbf{X} \) are the expressions multiplying \( dy \) and \( dx \) in the relation (7), respectively, being constants inside an element with a linear variation of displacements, \( \mathbf{V}_{K,j} \) and \( \mathbf{X}_{K,j} \) are the differentials of the coordinates of points \( K \) and \( J \) along the integration path and \( N \) is a number of elements along the integration path. This technique was tested (see Berković, 1980) in the case of a centrally cracked plate using eight different paths. The obtained results for J-integral varied only 1% from the mean value.

In the case of linear elastic problem J-integral reduces to the strain energy release rate and can be directly related with the stress intensity factor. In the case of the plane problem the relation between them is:

\[ \kappa_1 = \frac{\sqrt{2\pi}}{k_1 - J} \]  

(11)

RESULTS

The double edge cracked tension plate under plane strain condition was tested (Fig. 1). Due to the symmetry only a quarter of the plate was considered. The basic finite element mesh is presented at Fig. 2.
Five meshes were tested here, all of them obtained by adding one new "layer" of elements in the lower half of the domain. $K_I$ was calculated by the extrapolation technique and via $J$-integral. Results are given in Table 1.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>I</th>
<th>II</th>
<th>III</th>
<th>IV</th>
<th>V</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of elements N</td>
<td>32</td>
<td>44</td>
<td>60</td>
<td>84</td>
<td>108</td>
</tr>
<tr>
<td>Strain energy $U$</td>
<td>2.593</td>
<td>2.712</td>
<td>2.780</td>
<td>2.825</td>
<td>2.842</td>
</tr>
<tr>
<td>$K_I$ (extrapolation)</td>
<td>1.27</td>
<td>1.38</td>
<td>1.45</td>
<td>1.50</td>
<td>1.52</td>
</tr>
<tr>
<td>J</td>
<td>1.92</td>
<td>2.18</td>
<td>2.30</td>
<td>2.36</td>
<td>2.40</td>
</tr>
<tr>
<td>$K_I$ (via J)</td>
<td>1.45</td>
<td>1.55</td>
<td>1.59</td>
<td>1.61</td>
<td>1.62</td>
</tr>
</tbody>
</table>

The exact value for the strain energy is given for a quarter of the plate (Baboška and Szabo, 1982) as $U=0.7342$. Hellen has reported $K_I=1.669$ using virtual crack extension method (Hellen, 1973).

The largest computational error belongs to the fifth mesh. According to eqn (3), for $N=108$ and $h_{max}/h_{min}=45.5$, using single precision ($s=7.2$), the computational error is:

$$\text{ERROR} = c_1 \times 10^{-7} \times 45.5 \times 108 = 5 \times 10^{-4}$$

**DISCUSSION**

It is clear that relatively small number of elements give an accurate prediction for both the strain energy and stress intensity factor evaluated via $J$-integral. The extrapolation technique was not so successful. It is also clear that the single precision is sufficient for this type of a problem. If the more complicated problems should be solved, with much larger number of elements and ratio $h_{max}/h_{min}$ the double precision is to be used. This situation can be avoided using higher order elements, but they require the more complicate and less accurate relations for $J$-integral evaluation.

**CONCLUSIONS**

Rapid mesh refinement is very simple and effective technique for the evaluation of fracture mechanics parameters. The formulation itself is not limited regarding material properties - it is therefore possible to treat linear elastic, nonlinear elastic and elasto-plastic behaviour. However the care is to be taken with $J$-integral evaluation for elasto-plastic behaviour, since its path independency is doubtful in that case. Numerical experiments indicate large error of $J$-integral along the paths close to the crack tip, but the average value is in good agreement with predicted values (Shiratori and Miyoshi, 1980). Many authors have tried to reformulate $J$-integral in order to regain the path independency (Hellen, 1980). It is also suggested to simulate the elasto-plastic behaviour by an equivalent nonlinear elastic behaviour. This is satisfactory provided that no energy release rate is attached to.

$J$-integral (Knott, 1980).

The possibility of using simple, standard programme with the classical elements is a great benefit of this technique. Commercially available programme is used here (Hinton and Owen, 1977), modified only to accept the simplex triangular isoparametric elements. In the case of nonlinear elastic or elasto-plastic behaviour an excellent extension is the other book of the same authors (Hinton and Owen, 1980).

**REFERENCES**


