ELASTIC INTERACTION OF A CRACK WITH MICROCRACKS

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ABSTRACT

Elastic interactions of a crack with an array of microcracks located near the tip and imitating "damage" is analyzed. The stress field and the effective stress intensity factor are considered based on the double layer potential technique (known also as representation of cracks by dislocations) and polynomial conservation theorem. Two examples are given: (1) one microcrack on continuation of the crack line (stress "amplification") and (2) two microcracks parallel to the main crack and located at certain distance from the crack line; in this case, both stress "amplification" and stress "shielding" are possible depending on the microcrack's location.

KEYWORDS

Crack tip, double layer potential, second Green's tensor, interaction, microcracks, polynomial approximation, stress amplification, stress intensity factor, stress shielding.

INTRODUCTION

Displacement field generated by a single crack can be represented in the form of the double layer potential integral, with crack opening displacement (COD) being the potential density ("representation of crack by dislocations")

$$\underline{u}(\underline{x}) = \int_{\Omega} \underline{b}(\underline{x}') \cdot \underline{\phi}(\underline{x}', \underline{x}) d\underline{x}' \tag{1}$$

Where Ω is a crack surface (line, in a two-dimensional case) and $\underline{\phi}$ is the second Green's tensor. This integral exists only in the principal value sense. The stress field generated by a crack is:

$$\sigma(x) = \underline{T}_{x} \int_{\Omega} \underline{b}(x') \cdot \underline{\phi}(\underline{x}', \underline{x}) d\underline{x}' \tag{2}$$

where $\underline{T}_x$ denotes the stress operator transforming $\underline{u}(\underline{x})$ into a stress field; index "x" indicates that differentiation in $\underline{T}$ is to be performed with respect to $\underline{x}$. Note that representations (1) and (2) are valid for both two-and three-dimensional problems.

In the present paper, we consider a two-dimensional (plane stress) configuration consisting of a main crack (macrocrack) and N much smaller cracks (microcracks) located near the macrocrack tip. For simplicity of calculations, mode I remote loading of the main crack is assumed. The system of singular integral equations expressing traction-free boundary conditions on the crack faces describe the problem (Isida,1970;Barr and Clearly,1983). The essence of the proposed method is that the functions $\underline{b}_i$ are sought in the form (ellipse) x (polynomial) where the first multiplier corresponds to COD of a crack embedded into a uniform stress field and the second multiplier accounts for the nonuniformity of the stress field. Such representation is based on the following: (1) approximation of the stress field along the line of a given crack by polynomials and (2) theorem on polynomial conservation stating that the COD of a crack embedded into a polynomial stress field of degree N has the form (ellipse) x (polynomial of degree N). Using these we reduce the system of integral equations to a system of linear algebraic equations for the polynomials' coefficients. In many cases the use of linear polynomials (i.e., representation of COD by "linearly distorted" ellipses) is sufficient. When the obtained system is large and inconvenient for the direct solution, an iterative approach is proposed (Chudnovsky and Kachanov, 1983). The iterations have clear physical interpretations: the zeroth iteration approximates the stress field by the macrocrack tip dominated field, the first iteration gives contributions from non-interacting microcracks embedded into the macrocrack tip field, the next iterations account for the first, double and higher order interactions.

FORMULATION OF THE PROBLEM

Stress field near the macrocrack tip is a superposition

$$\underline{\sigma}(\underline{x}) = \underline{\sigma}^{\infty} + \underline{\hat{\sigma}}(\underline{x}) + \sum_{i=1}^{N} \underline{\sigma}_i(\underline{x}) \tag{3}$$

where $\underline{\sigma}^{\infty}$ is the stress field due to remotely applied loads in the absence of cracks. $\underline{\hat{\sigma}}$ and $\underline{\sigma}_i$ are the stress fields generated by the main crack and by the i-th microcrack, respectively. Near the macrocrack tip stresses can be neglected compared to $\underline{\hat{\sigma}}$. The field $\underline{\hat{\sigma}}$ can be represented as $K_I^{eff} \cdot \underline{\sigma}_o(\underline{x})$ where $\underline{\sigma}_o = \underline{\phi}(\theta)/\sqrt{2\pi r}$ denotes the "standard" mode I crack tip field. Thus,

$$\underline{\sigma}(\underline{x}) = K_I^{eff}\underline{\sigma}_o(\underline{x}) + \sum_{i=1}^{N} \underline{\sigma}_i(\underline{x}) = K_I^{eff}\underline{\sigma}_o(\underline{x}) + \sum_{i=1}^{N} \underline{T}_x \int_{\Omega_i} \underline{b}_i(\xi_i)\cdot\underline{\phi}(\xi_i,\underline{x})\,d\xi_i \tag{4}$$

where ξ_i is a coordinate along the i-th crack. Expression (4) contains 2N+1 scalar unknowns: components of N vector functions $\underline{b}_i(\xi)$ and K_I^{eff}. Traction free boundary conditions on the microcracks result in N vectorial equations:

$$\underline{n}_i\cdot\left\{K_I^{eff}\underline{\sigma}_o(\underline{x}) + \sum_{\substack{k=1\\k\neq i}}^{N} \underline{T}_x \int_{\Omega_k} \underline{b}_k(\xi_k)\cdot\underline{\phi}(\xi_k,\underline{x})\,d\xi_k + \underline{T}_x \int_{\Omega_i} \underline{b}_i(\xi_i)\cdot\underline{\phi}(\xi_i,\underline{x})\,d\xi'_i\right\} = 0,\quad \underline{x}\in\Omega_i;\ i=1,\dots,N \tag{5}$$

The last integral in braces converges in the principal value sense; it becomes divergent, if the stress operator is moved under the integral and applied directly to $\underline{\phi}(\underline{\xi}, \underline{x})$. It can be shown, however,that its limit value is given by the following regularization (Kanaun, 1974):

$$\lim_{\underline{x}\to\xi_i} \underline{T}_x \int_{\Omega_i} \underline{b}_i(\xi'_i)\cdot\underline{\phi}(\xi'_i\ \underline{x})\,d\xi'_i = \int_{\Omega_i} [\underline{b}_i(\xi'_i)-\underline{b}_i(\xi_i)]\cdot\underline{T}_x[\underline{\phi}(\xi'_i,x_i)]\,d\underline{\xi}' \tag{6}$$

where the integral on the right converges in the principal value sense.

Expression (4) contains one more unknown, K_I^{eff}. An additional equation represents boundary condition on the main crack $(-\ell_o, \ell_o)$ with unit normal $\underline{n}$ and reflects the impact of the microcrack array on the main crack:

$$K_I^{eff} = K_I^o + \frac{1}{\sqrt{\pi\ell_o}}\int_{-\ell_o}^{\ell_o}\sqrt{\frac{\ell+\xi}{\ell-\xi}}\ \underline{n}\cdot\sum_{i=1}^{N}\underline{\sigma}_i(\xi)\cdot\underline{n}\,d\xi \tag{7}$$

Two particular crack configurations are considered below.

ONE MICROCRACK ALLIGNED WITH A MACROCRACK, PIECEWISE CONSTANT APPROXIMATION (Fig. 1)

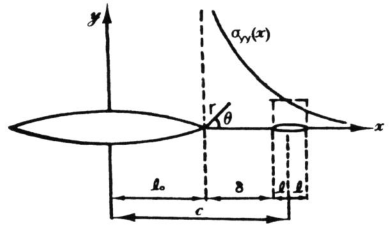

FIGURE 1

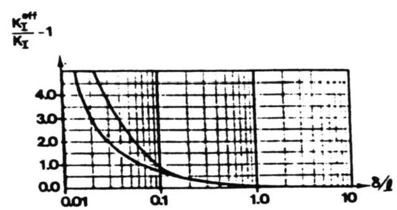

FIGURE 2

It is assumed here that the stress σ_{yy} induced by the main crack tip field along the microcrack line does not change much within this line and can, therefore, be approximated by a constant, taken to be $\sigma_{yy}(c)$ (see Fig. 1). (This approximation becomes inadequate if the microcrack is located very close to the main crack tip; this case is considered in the last section). The COD of the microcrack is elliptic and is related to a uniform traction $\sigma_{yy}(c)$ by the formula:

$$b_y(\xi) = \frac{4\ell}{E}\,e(\xi)\,\sigma_{yy}(c) \tag{8}$$

where ξ is a coordinate along the crack counted from its center (Fig. 1) $e(\xi) = \sqrt{1-[(\xi-c)/\ell]^2}$ and E is Young's modulus. Boundary condition (5) simply means that traction generating the COD (8) is the one induced by the K_I^{eff} - dominated field along $(c-\ell, c+\ell)$, i.e., in the framework of piecewise constant approximation, $\sigma_{yy} = \sigma_{yy}(c) = K_I^{eff}/\sqrt{2\pi(\ell+\delta)}$. Equation (7) yields

$$K_I^{eff} = K_I^o + \frac{1}{\sqrt{\pi \ell_o}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + t}{\ell_o - t}}\, \sigma_{yy}^{\ell}(t)\,dt$$

where σ_{yy} is the stress generated by the microcrack embedded into the K_I^{eff} - dominated field and to be found from (2) with (8) as the potential density. Calculating the integrals involved in (7) and (2) we obtain:

$$K_I^{eff} = K_I^o + K_I^{eff}\, q(\tfrac{\delta}{\ell}) \quad \text{so that} \quad K_I^{eff} = K_I^o/(1-q) \qquad (9)$$

where

$$q = q(\tfrac{\delta}{\ell}) = \frac{1}{\sqrt{2(\ell'+\delta')}} \int_{-1}^{1} \sqrt{\frac{1+t}{1-t}} \left[\frac{1}{\sqrt{1-(\frac{\ell}{t-c})^2}} - 1 \right] dt \qquad (10)$$

and $\ell' = \ell/\ell_o$, $\delta' = \delta/\ell_o$. The graphs of K_I^{eff}/K_I^o are shown in Fig. 2 (lower curve). $K_I^{eff} \to \infty$ when the distance between two cracks tends to zero.

The stress field is given by the formula:

$$\underline{\sigma}(\underline{x}) = K_I^{eff} \left\{ \frac{\underline{\Phi}[\theta(\underline{x})]}{\sqrt{2\pi r(\underline{x})}} + \frac{4\ell}{\sqrt{2\pi(\ell+\delta)}\cdot E}\, T_x \int_{c-\ell}^{c+\ell} e(\xi)\,\underline{n}\cdot\underline{\Phi}(\xi,\underline{x})\,d\xi \right\} \qquad (11)$$

The first term in braces represents the main crack tip field, the second term represents the stress field generated by the microcrack.

One comment should be made with respect to the solution obtained. In the problem considered only one microcrack was involved and the system of equations (5), (7) was reduced to one equation for K^{eff} which permits the exact (in the framework of piecewise constant approximation) solution. In problems involving many microcracks, however, the system of linear algebraic equations for the polynomials' coefficients is large and may be inconvenient for analytical solution. A method of approximate solution has been proposed (Chudnovsky and Kachanov, 1983). It is based on iterations corresponding to multiple crack interactions; their physical meaning can be demonstrated in two crack interaction problem. Solving equation $K_I^{eff} = K_I^o + qK_I^{eff}$ by iterations we obtain $K_I^{eff} = K_I^o\,(1+q+q^2+...)$; the sum of this series converges to (9) for $q<1$. In the sequence of iteration terms, the first term gives the macrocrack tip field in the absence of microcracks, the second term accounts for the first order interaction, i.e., the stress field generated by the microcrack embedded into the K_I^o-dominated field gives correction $K_I^o q$. The third term accounts for the second correction, etc. (Fig. 3). Such a solution with

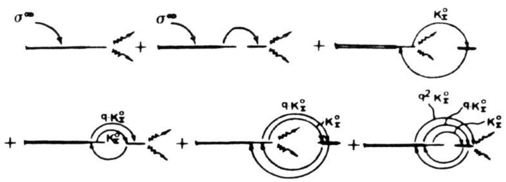

FIGURE 3

only a finite number of terms retained gives an approximation in two different senses: piecewise constant approximation of the stresses along the microcrack line and an approximation by the multiplicity of crack interactions taken into account. Note that such iterative procedure is particularly convenient if geometry of the microcrack array is known only in probabilistic terms (Chudnovsky and Kachanov, 1983).

TWO MICROCRACKS PARALLEL TO THE MAIN CRACKS (Fig. 4)

In this configuration two different results of crack interaction may occur depending on the relative values of the geometrical parameters; one when $\delta >> \ell_o$

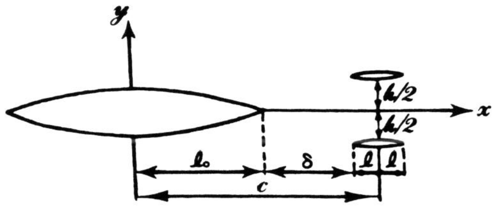

FIGURE 4

and the microcracks "amplify" the stress concentration ($K_I^{eff} > K_I^o$), the other when $\delta \approx \ell_o$ and the microcacks "shield" the macrocrack tip ($K_I^{eff} < K_I^o$). In the intermediate range these two mechanisms compete.

Because of the symmetry of the problem there are no mode II terms and equations (5) take the form

$$\begin{aligned} \sigma_{yy}(c,\tfrac{h}{2}) + f_{21}\,\sigma_{yy}(c,-\tfrac{h}{2}) &= -K_I^{eff}\, \frac{\Phi_{yy}[\theta(c,\frac{h}{2})]}{\sqrt{2\pi r(c,\frac{h}{2})}} \\ \sigma_{yy}(c,\tfrac{h}{2}) + f_{12}\,\sigma_{yy}(c,\tfrac{h}{2}) &= -K_I^{eff}\, \frac{\Phi_{yy}[\theta(c,-\frac{h}{2})]}{\sqrt{2\pi r(c,-\frac{h}{2})}} \end{aligned} \qquad (12)$$

The first equation expresses a boundary condition on the microcrack ℓ_1. The first term of it represents the normal stress at the center of the microcrack ℓ_1, the second term represents the stress component σ_{yy} exerted on ℓ_1 by ℓ_2 with f_{21} being an influence function and the right hand term is the dominating stress field at the center of ℓ_1. The equation is formulated at the center of ℓ_1 because of the piecewise constant approximation. The second equation is formulated for the microcrack ℓ_2 and, because of symmetry, it is identical to the first one. The influence function $f_{12} = f_{21} \equiv f$ is expressed in terms of the double layer potential:

$$f = f(\ell,\underline{x}) = \frac{4\ell}{E}\,(\underline{n}\;\underline{n}):T_x \int_{\ell} e(\xi)\,\underline{n}\cdot\underline{\Phi}(\xi,\underline{x})\,d\xi \qquad (13)$$

where $\ell = \ell_1 = \ell_2$ and $\underline{n}$ is a unit normal to the microcracks. An additional equation for K_I^{eff} is:

$$K_I^{eff} = K_I^o + \frac{2}{\sqrt{\ell\pi_o}} \int_{-\ell_o}^{\ell_o} \sigma_{yy}^{\ell}(t) \sqrt{\frac{\ell_o + t}{\ell_o - t}}\, dt \tag{14}$$

where σ_{yy} is related to $\sigma(c, \frac{h}{2})$ by the following formula:

$$\sigma_{yy}^{\ell}(\underline{x}) = \frac{4\ell}{E} \sigma(c, \frac{h}{2})\ \underline{n}\,\underline{n} : \underline{T}_x \int_{\ell} e(\xi)\underline{n}\cdot\underline{\Phi}(\xi, \underline{x})\, d\xi \tag{15}$$

Thus, (14) and one of the equations (12) constitute a system of two linear algebraic equations for two unknowns $\sigma_{yy}(c, \frac{h}{2})$ and k_I^{eff}. The solution for K_I^{eff} has the same form (9), with,

$$q = - \frac{2}{\sqrt{\pi\ell_o}[1+f(\ell,h)]} \frac{\phi_{yy}[\sigma(c,\frac{h}{2})]}{\sqrt{2\pi r(c,\frac{h}{2})}} \int_{-\ell_o}^{\ell_o} \sqrt{\frac{\ell_o + t}{\ell_o - t}}\, f(\ell, t)\, dt \tag{16}$$

In order to analyze the behavior of K_I^{eff} as a function of geometrical parameters, the influence function must be evaluated. The estimates of $f(\ell, x)$ for large and small ℓ/h can be easily obtained. Using these estimates q can be represented in the form

$$q = \begin{cases} -\frac{3}{n} \cdot \frac{6}{1+6(\ell/h)^2} \cdot \left(\frac{\ell}{h/2}\right)^2 & \text{for small } \ell/h \\ -\sqrt{\frac{2+\ell'}{2}} \cdot \frac{\ell}{h} & \text{for large } \ell/h \end{cases} \tag{17}$$

Formulas (17) give negative values of q which according to (9) gives $K_I^{eff} < K_I^o$ which demonstrates the presence of "shielding" effect. The graph of K_I^{eff}/K_1^o vs. $2\ell/\frac{h}{2}$ is shown in Fig. 5.

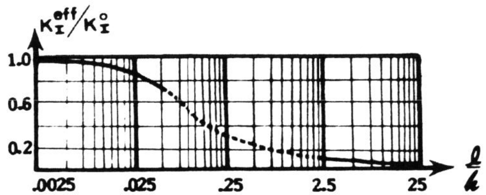

FIGURE 5

LINEAR AND HIGHER ORDER APPROXIMATIONS

One microcrack aligned with a macrocrack was considered above on the basis of piecewise constant approximation of stresses on the microcrack. In this section we show how the variability of stresses along the microcrack can be taken into account.

In a linear polynomial approximation, the tractions along the microcrack line are assumed to be of the form:

$$\underline{n}\cdot\underline{\sigma}(x') = \underline{n}\cdot\underline{\sigma}(c') + \underline{n}\cdot\underline{\sigma}'(c')(x'-c'), \qquad x' \epsilon (c'-\ell', c'+\ell') \tag{18}$$

where $x' = x/\ell_o$, $c' = c/\ell_o$ and $\underline{\sigma}'(c')$ is a derivative with respect to x' taken at the microcrack center. According to the polynomial conservation theorem (Willis, 1968) the microcrack COD can be represented in the form of a "linearly distorted" ellipse:

$$\underline{b}(\xi) = [b_o + b_1(\xi'-c')]\ 4e(\xi)\underline{n} \tag{19}$$

where b_o and b_1 are some unknown coefficients. The displacement field generated by the microcrack is

$$\underline{u}(\underline{x}) = \int_{c-\ell}^{c+\ell} [b_o + b_1(\xi-c)]4e(\xi)\underline{n}\cdot\underline{\Phi}(\xi, \underline{x})\, d \tag{20}$$

Application of the stress operator $\underline{T}_x$, substitution of the Green's function and evaluation of the integral result in the following expression for the traction σ_{yy} along the microcrack line $(c-\ell, c+\ell)$: $\sigma_{yy}^{\ell}(x') = (E/\ell)[b_o + 2b_1(x'-c')]$ and, comparing coefficients of the linear functions, we obtain $\sigma_{yy}(c') = (E/\ell)b_o$, $\sigma'_{yy}(c') = 2(E/\ell)b$. Following the same line of reasoning for the polynomial tractions on the microcrack

$$\underline{n}\cdot\underline{\sigma}(x') = \sum_K^P \underline{n}\cdot\underline{\sigma}^{(K)}(c') \frac{(x'-c')^K}{K!} \tag{21}$$

(where $\sigma^{(K)}(c')$ is the k-th derivative in the direction of the microcrack taken at its center) the set of coefficients b_k can be found. The vector column b_k appears in the expression of a crack opening displacement which is the generalization of (19):

$$\underline{b}(\xi) = \sum_K^P \underline{n}\ b_K e(\xi) \frac{(x'-c')^K}{K!} \tag{22}$$

Two vector columns $\underline{n}.\sigma^{(k)}(c) = \{\sigma\}$ and $b_k = \{b\}$ are linearly related through the matrix $\{A\}$: $\{b\} = \{A\}\{\sigma\}$. It can be shown that the matrix $\{A\}$ can be represented in the form

$$\{A\} = \frac{\ell}{E} \begin{pmatrix} 1 & 0 & \beta_{02}\ell'^2 & 0 & \beta_{06}\ell'^4 & 0 & \beta_{06}\ell'^6 & . & . & . \\ 0 & \frac{1}{2} & 0 & \beta_{04}\ell'^2 & 0 & \beta_{15}\ell'^4 & 0 & . & . & . \\ 0 & 0 & \frac{1}{3} & 0 & \beta_{24}\ell'^2 & 0 & \beta_{26}\ell'^4 & . & . & . \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & \frac{1}{P} \end{pmatrix} \quad (23)$$

Where the coefficients β_{ik} in every row form the decreasing sequences. The components of the vector column $\{\sigma\}$ must be determined from the boundary condition on the microcrack:

$$\sigma_{yy}(c) = K_I^{eff} \underline{\sigma}_{oyy} (\ell + \delta), \quad (24)$$

and the relatiaons which follow from the latter:

$$\sigma'_{yy}(c) = K_I^{eff} \sigma'_{oyy}(\ell + \delta), \qquad \sigma''_{yy}(c) = K_I^{eff} \sigma''_{oyy}(\ell + \delta), \quad (25)$$

$$\ldots, \qquad \sigma^{(P)}_{yy}(c) = K_I^{eff} \sigma^{(P)}_{oyy}(\ell + \delta)$$

Substituting (24) and (25) into the linear form, with the use of (4) we obtain the resulting stress field in the form:

$$\underline{\sigma}(\underline{x}) = K_I^{eff} \frac{\underline{\phi}[\Theta(\underline{x})]}{\sqrt{2\pi r(\underline{x})}} + \frac{1}{\pi} \sum_{K=0}^{P} \sum_{n=0}^{N} A_{nk} \sigma^{(K)}_{oyy}(\ell+\delta) \, \underline{T}_x \int_{c-\ell}^{c+\ell} \frac{(\xi-c)^n}{n!} e(\xi) \underline{n}\cdot\underline{\Phi}(\xi,\underline{x}) d\xi \quad (26)$$

where K_I^{eff} has the form (9) again. Using this one can evaluate an approximate solution. For instance the upper curve of Fig. 2 represents the graph of K_I^{eff} as given by the linear approximation; it shows that, unless δ/ℓ becomes very small, the results given by the linear and by the piecewise constant approximations are close.

It can be shown that the series converge when $P \to \infty$. Note in conclusion, that the procedure described above can be applied to the macrocrack surrounded by the array of microcracks of aribtrary configuration.

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