ANALYTICAL SOLUTION OF PLASTIC STRAIN CONCENTRATION (PROOF OF NEUBER'S EMPIRICAL EQUATION)

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INTRODUCTION

Analytical solution of plastic strain concentration is a problem that has remained unsolved for many years. It was believed to be too complicated. At present, the only way out is by computer-aided numerical solutions which cannot substitute for analytical solutions.

In fact, plastic strain concentration turns out to be much simpler than expected. It can be calculated from elastic strain field using the rule of sliding, contrary to the common belief that plasticity cannot be calculated from elasticity.

The rule of sliding was calculated from tensile stress (Anon, 1977). The tensile stress specimen elongates freely without hindrance, therefore the rule of sliding obtained from it is associated with free sliding. It happens that the plastic strain concentration factor is not affected by plastic deformation, so there is nothing to interfere with the sliding; therefore the rule of free sliding from tensile test diagram is anplicable.

For verification, Neuber's equation is used. Neuber's equation, when solved together with the tensile test diagram, yields an empirical solution of plastic strain concentration and the result is approximately the same as the analytical solution mentioned above, only with a larger error than the analytical solution as compared with experimental data.

However, it is worthwhile to point out that in the ASME-Boiler Code it is recommended that the imaginary elastic stress (higher than ultimate stress) be used to calculate, approximately, the fatigue life of nozzles by fatigue curve. This imaginary elastic stress is the same as that calculated by K_e in Neuber's equation, and therefore, this imaginary elastic stress has been used empirically and successfully for more than 20 years.

NONHOMOGENEITY OF MATERIAL AND SOLUTION BY RELAXATION

The analytical solution of plastic strain concentration is a problem which has remained unsolved for more than half of a century. The solution is believed to be too complicated because of the fact that the stress-strain law during plastic deformation depends upon history of loading.

Fortunately, a technique is available which can be used to avoid these difficulties. It consists in resolving the stress above the yield value into two components - elastic and ideal plastic stresses - which can be calculated separately and more easily.

This resolution of plastic stress is not new. Forty years ago, Timoshenko (1946) used it to explain strain hardening (see Appendix). Accordingly, in order to obtain the elastic component of stress σ^* from plastic stress σ Eq. (8) in the Appendix may be used:

$$\sigma^*/\sigma_s - 1 = m'(\sigma/\sigma_s - 1) \quad (1)$$

where m' = m is usually called strain hardening index.

In the case of simple strain concentration problems such as for elliptical holes, notches, etc., σ^* will be the elastic stress calculated by ordinary theory of elasticity. This can be illustrated by the idea of relaxation as follows:

(1) To illustrate that the stress concentration factor for ellipses do not appreciably change because of plastic deformation, let us take a 40 x 29 mm ellipse as an example. The increase of short diameter will be approximately $\pi/4\ P\varepsilon_{max}$ where $P = b^2/a = 10^2/20 = 5$ mm. If $\varepsilon_{max} = 20\%$, then short diameter will increase less than 1 mm. Thus, the stress concentration factor will change from originally 5 to $1 + 2 \times 40/21 = 4.81$ which is only 4% and, hence, is negligible.

(2) To derive that σ^* equals (elastic) concentration stress, the relaxation idea can be used. First, hold the plastic sliding fast so that the specimen can act only elastically. The plastic sliding is allowed to occur afterwards.

Since the stress concentration field does not change during plastic sliding (Anon, 1977), a portion of cross section may be assumed to slide freely while the other portion remains unchanged. This is just what happens according to relaxation idea in the free sliding in tensile test specimens where weakening and lengthening occur simultaneously. The rule of free sliding (see Eq. (6) in the Appendix) yields:

$$\varepsilon^*/\varepsilon - 1 = \log_2 \varepsilon/\varepsilon_s = 3.32 \log_{10} \varepsilon/\varepsilon_s \quad (2)$$

For practical materials, of course, there are some deviations from the theoretical coefficient 3.32 and these can be calculated directly from tensile test diagram.

EXPERIMENTAL CHECK AND MUTUAL PROOF WITH NEUBER'S EQUATION

In order to show the applicability of the technique, plastic strain concentrations at elliptical holes are measured (see Fig. 1) as shown in Fig. 2, which agree with theoretical curve (hard duraluminum) quite well:

$$\sigma^*/\sigma_s - 1 = 2.46 \log_{10} \varepsilon/\varepsilon_s \quad (3)$$

The theoretical curve is calculated from measured tensile test diagram as follows:

(1) Plot the tensile test diagram on log-log paper using dimensionless coordinates of stress σ/σ_s and of strain $\varepsilon/\varepsilon_s$.

(2) Draw a straight line (or lines) through point ($\sigma/\sigma_s=1$, $\varepsilon/\varepsilon_s=1$) to represent the tensile test curve approximately, and measure the inclination m.

(3) Calculate the elastic stress according to:

$$\sigma^*/\sigma_s - 1 = m(\sigma/\sigma_s - 1) \quad (4)$$

(4) Plot the $\log\varepsilon/\varepsilon_s$ - $(\sigma^*/\sigma_s - 1)$ relation on semilog paper and measure the inclination C to obtain the equation:

$$\sigma^*/\sigma_s - 1 = C \log_{10} \varepsilon/\varepsilon_s \quad (5)$$

For verification, Neuber's Equation is used. Neuber's Equation $(K_e)^2 = K_p Q_p$, in terms of external stress, σ_s, can be written as follows:

$$(\sigma^*_{max}/\sigma_s)^2 = \sigma_{max}/\sigma_s \times \varepsilon_{max}/\varepsilon_s . \quad (6)$$

An alternative for any σ and ε is:

$$(\sigma^*_{max}/\sigma_s)^2 = \sigma/\sigma_s \times \varepsilon/\varepsilon_s \quad (7)$$

where σ^* denotes stress calculated by elasticity, as it may be much higher than σ in tensile test diagram which can be $2\sigma_s$ at most, while σ^*_{max} can be 10 when $\varepsilon/\varepsilon_s \simeq 100$.

Neuber's Equation is solved together with tensile test diagram to eliminate σ/σ_s. We then obtain a σ^*/σ_s - $\varepsilon/\varepsilon_s$ relation shown in Fig. 2 by a dashed line.

Neuber's Equation is an approximate empirical equation, and in fact the above mentioned elliptical hole experiments are a repetition of Neuber's experiments. The present method yields results which agree approximately with Neuber's Equation, as shown in Fig. 2.

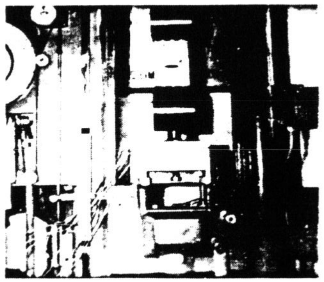

Fig. 1. Plastic strain concentration at elliptical hole.

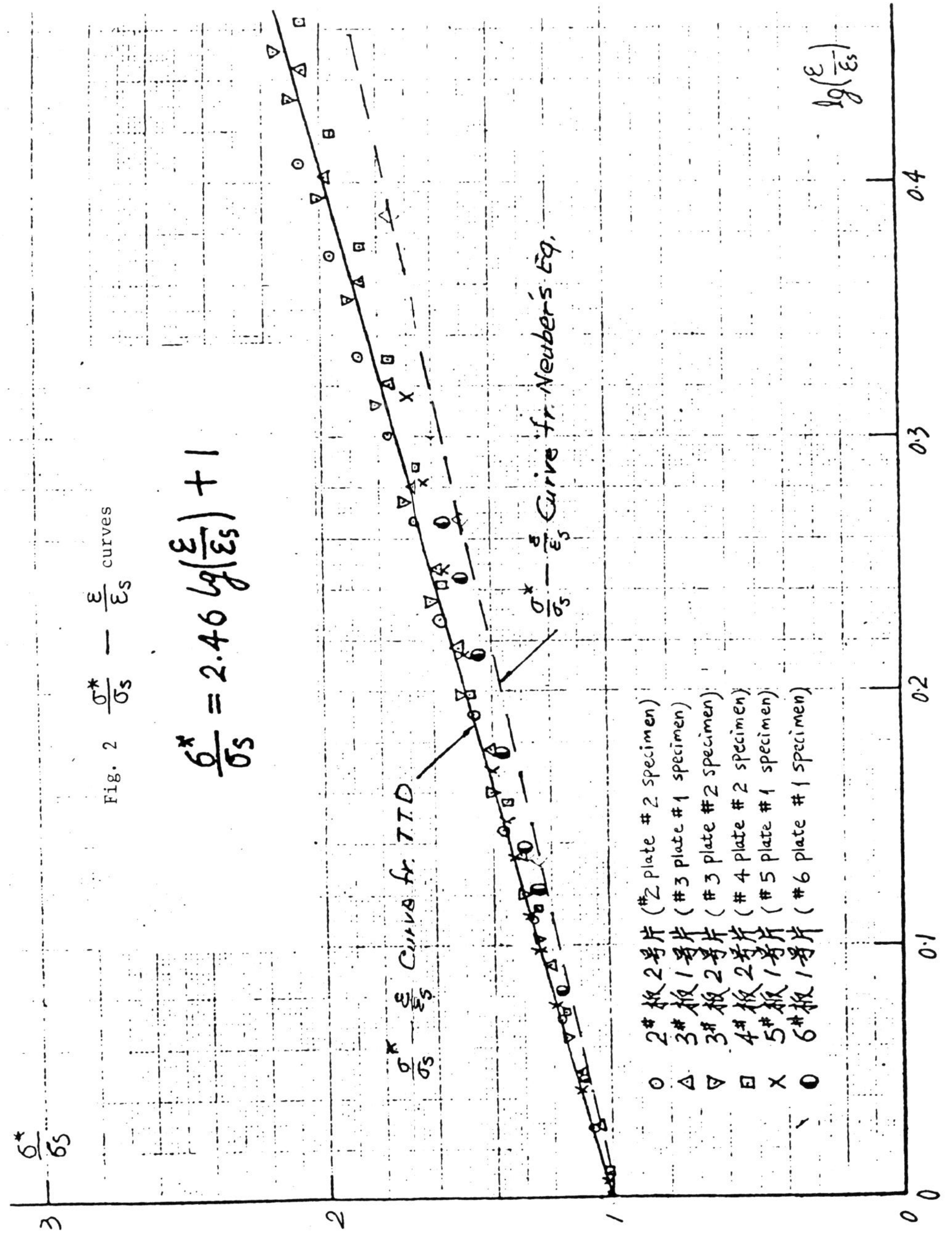

Fig. 2 $\frac{\sigma^*}{\sigma_s} - \frac{\varepsilon}{\varepsilon_s}$ curves

APPENDIX

Rule of Sliding:
Tensile test diagram can be approximately represented by exponential curves as:

$$\varepsilon/\varepsilon_s = (\sigma/\sigma_s)^m \quad \text{or} \quad \log_2 \varepsilon/\varepsilon_s = m \log_2 \sigma/\sigma_s \qquad \text{(A1)}$$

Exponential curves of several carbon steels are plotted in Fig. 3 by dashed lines.

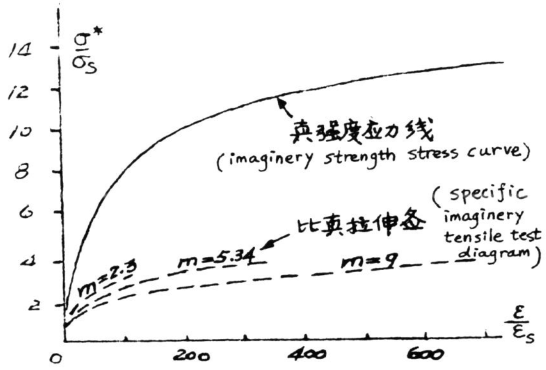

Fig. 3. Exponential curves of several carbon steels.

When plastic sliding occurs after yield point, the steepness of the curve drops from elastic so that the specific stress σ/σ_s decreases and specific strain $\varepsilon/\varepsilon_s$ increases. Yet the weakening and elongating effects of plastic sliding are different for different steels.

Notice that there is only one variable m to denote the difference of plasticity while the plastic sliding effects are two--weakening and elongating, therefore if we fortunately can eliminate m by transformation, then the abscissa transformation will be free from m, i.e., the same for all steels. The two-step transformations as a whole eliminates m, so the effect of plastic sliding is to restore the elastic relation:

$$\varepsilon/\varepsilon_s = (\sigma/\sigma_s)^1$$

The two-step transformation is as follows:

(1) Use ordinate transformation to eliminate weakening.
By comparison of $\sigma/\sigma_s - 1$ & $\log_2 \sigma/\sigma_s$ in Table 1 for range of $\sigma/\sigma_s = 1 \sim 2$ of tensile test diagrams for ordinary steels,

TABLE 1 Comparison of $\sigma/\sigma_s - 1$ & $\log_2 \sigma/\sigma_s$ of Tensile Test Diagrams for Ordinary Steel.

σ/σ_s	=	1.0	1.2	1.4	1.6	1.8	2.0
σ/σ_s	=	0.0	0.2	0.4	0.6	0.8	1.0
$\log_2 \sigma/\sigma_s$	=	0.0	0.26	0.48	0.68	0.85	1.0

it can be seen that, approximately:

$$\sigma/\sigma_s - 1 \approx \log_2 \sigma/\sigma_s \qquad \text{(A2)}$$

Therefore, equation (A1) becomes:

$$\log_2 \varepsilon/\varepsilon_s \approx m\ (\sigma/\sigma_s - 1) \qquad \text{(A3)}$$

Now let us transform σ to stress σ^* so that the plastic part of σ above σ_s is raised m times, i.e.:

$$m\ (\sigma/\sigma_s - 1) = \sigma^*/\sigma_s - 1 \qquad \text{(A4)}$$

Then, from equations (A1) and (A2), we obtain:

$$\sigma^*/\sigma_s - 1 = \log_2 \varepsilon/\varepsilon_s \qquad \text{(A5)}$$

This transformation not only eliminates m while the abscissa remains untouched, but it also satisfies the transition condition at yield point requiring continuity.

(2) Use abscissa transformation to obtain elastic line. From equation (A5) it can be seen that if we transform to its linear scale, $\varepsilon^*/\varepsilon_s - 1$, as in equation (A6), then elastic line will be obtained as equation (A7):

$$\varepsilon^*/\varepsilon_s - 1 = \log_2 \varepsilon/\varepsilon_s \qquad \text{(A6)}$$

$$\varepsilon^*/\varepsilon_s - 1 = \sigma^*/\sigma_s - 1 \qquad \text{(A7)}$$

Thus, equation (A6) is the required Rule of Plastic Sliding. To understand the physical meaning of this transformation, the strain hardening mechanism Timoshenko (1946) introduced may be used. An explanation follows.

Owing to grain orientation (and, of course, other nonhomogeneities) a portion of material, say 1/m' of the total cross section, yields while the rest is elastic. Therefore, the portion of load above yield point is solely carried by the elastic portion; thus the apparent stress, assuming the whole cross-section carries the load, will be only 1/m' of the elastic stress existing in the elastic portion of the material. Now that the external load is released, both the elastic and plastic portions will recover elastically and create internal stresses at equilibrium.

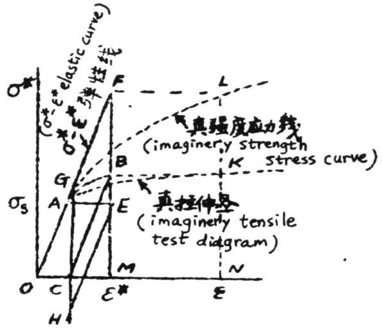

Fig. 4. Strain hardening mechanism.

Now if we want to calculate the elastic stress σ^*/σ_s from the apparent stress σ/σ_s, we simply multiply $\sigma/\sigma_s - 1$ by m', i.e.:

$$\sigma^*/\sigma_s - 1 = m'\ (\sigma/\sigma_s - 1) \qquad \text{(A8)}$$

Compare Equations (8) and (4) we see that the new elastic stress obtained by 1st step ordinate transformation is the elastic stress in the elastic portion of material and the strain hardening index m equals m', the ratio of area of plastic portion to the total cross-section.

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