ANALYTICAL AND NUMERICAL SOLUTIONS FOR CONFIGURATIONS MODELING FINITE CRACKED PLATES UNDER HARMONIC LOADING

V. Z. Parton and V. G. Boriskovsky

Moscow Institute of Chemical Engineering, USSR

ABSTRACT

A numerical method for the determination of the dynamic stress intensity factors in finite cracked bodies under harmonic loading is proposed. Analytical and numerical results for a cracked strip and cracked plates are listed.

KEYWORDS

Numerical methods, stress intensity factors, cracks, fracture, harmonic loading.

INTRODUCTION

In recent years, interest in problems of the evaluation of dynamic stress intensity factors for cracks disposed at the half-plane or half-space boundary or approaching it has increased enormously (Boriskovsky and Parton, 1983). The reason for such an interest lies in the fact that in practice the proximity of the boundary to the fracture source has to be taken into account, especially in dynamic problems when a complicated wave field is generated due to multiple reflections.

The authors have investigated at different times a number of fracture mechanics dynamic problems for harmonic loading of cracked bodies, which give information about finite cracked plate vibrations. The problem for a cracked strip was solved by Parton (1972), Parton and Morozov (1978) using analytical methods, and the problems for a rectangular plate with an edge (Boriskovsky, 1979) and centre cracks were solved with the help of the numerical finite element method.

When solving the problem for the strip $-\infty < y < \infty$, $|x| \le L$ with a symmetrical (relative to y) crack of length 2ℓ with normal loading $q\exp(i\omega t)$(q = const), the dual trigonometric series method was used and the following results were obtained. The dependence of stress intensity factor $K_I/q\sqrt{\ell}$ on the crack length ℓ/L for different frequencies $\omega^* = 2\omega L/c_1\pi$ of loading is shown in Fig. 1 (c_1 is the P-wave velocity).

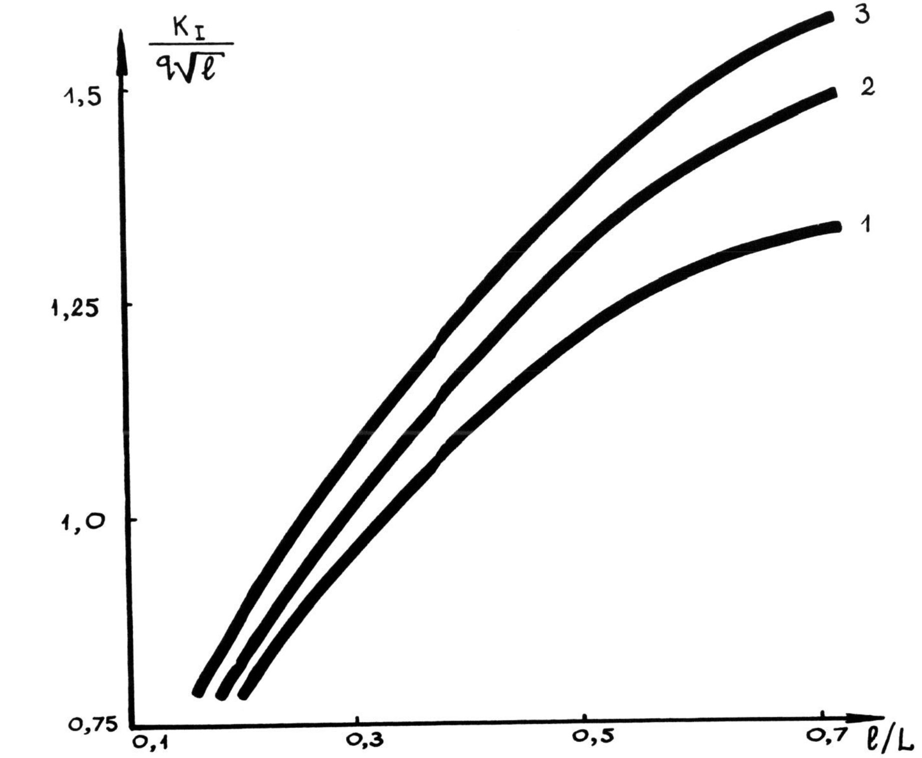

Fig. 1. The dependence of stress intensity factor on the crack length for different frequencies in the infinite cracked strip.

Line 1 corresponds to the static case $\omega^* = 0$; line 2 to $\omega^* = 0.274$ and line 3 to $\omega^* = 0.316$. These results demonstrate that over a wide range of frequencies, the inertial effect is connected with a decrease in fracture load level.

The numerical method for the determination of dynamic stress intensity factor in cracked plates under harmonic loading is based on the representation of these factors in the form of a superposition of "conditional" stress intensity factors corresponding to normalized vibrational modes with some weighting multipliers. The method proposed here seems to be more convenient for problems under consideration.

The formulation of basic equations

The finite element equations of motion of elastic body without damping under harmonic loading are given by

$$[M]\{\ddot{x}\} + [K]\{x\} = \{f\} e^{i\omega t}, \tag{1}$$

where $[M]$ is the mass matrix, $[K]$ the stiffness matrix, $\{x\}$ the displacement vector, and $\{f\}$ the loading vector.

For $\omega = 0$ we obtain the static equations

$$[K]\{x\} = \{f\} . \tag{2}$$

Denoting by ω_i^2 the eigenvalues in increasing order and by $\{x^{(i)}\}$ the eigenvectors from the general eigenvalue problem

$$[K]\{x\} = \omega^2 [M]\{x\} , \tag{3}$$

$$\{x^{(i)}\}^T [M] \{x^{(j)}\} = \delta_{ij} \tag{4}$$

we can write the particular solution with the perturbation frequency in the form

$$\{x(t)\} = \sum_i \{x^{(i)}\} \left(\{x^{(i)}\}^T \{f\}\right) (\omega_i^2 - \omega^2)^{-1} e^{i\omega t} \tag{5}$$

Hence, the static solution can also be represented in the form of a superposition of natural vibrational modes:

$$\{x^{(s)}\} = \sum_i \{x^{(i)}\}\left(\{x^{(i)}\}^T \{f\}\right) \omega_i^{-2} . \tag{6}$$

We now introduce the following definitions: K_s – the static stress intensity factor corresponding to $\{x^{(s)}\}$, $K^{(i)}$ – the "conditional" stress intensity factor, corresponding to $\{x^{(i)}\}$, and $K(t)$ – the dynamic stress intensity factor.

The dimensionality of $K^{(i)}$ is defined by taking Eq. (4) into account. The stress intensity factor can be determined in terms of the displacement vector by using a linear functional, so that

$$K(t) = \sum_i \left[\left(\{x^{(i)}\}^T \{f\}\right)(\omega_i^2 - \omega^2)^{-1} e^{i\omega t}\right] K^{(i)}, \tag{7}$$

$$K_s = \sum_i \left[\left(\{x^{(i)}\}^T \{f\}\right) \omega_i^{-2}\right] K^{(i)} . \tag{8}$$

Using the notation

$$z_i = \frac{K^{(i)}\left(\{x^{(i)}\}^T \{f\}\right)}{K_s\, \omega^2} , \tag{9}$$

we get from (6), (7)

$$K(t) = K_s \sum_i z_i \omega_i^2 (\omega_i^2 - \omega^2)^{-1} e^{i\omega t}, \tag{10}$$

$$\sum_i z_i = 1 . \tag{11}$$

The last equality can be used as a criterion for the determination of the accuracy of dynamic stress intensity factor if the static stress intensity factor K_s appearing in Eq. (10) is determined directly from the static Eq. (2) and not from (8). Besides, in the region $0 \leq \omega < \omega_1$ the required number of modes for (10) can be found from equality (11). Thus, the error in the determination of the dynamic stress intensity factor is estimated by the difference $|\sum z_i - 1|$. In other words, for the $K(t)$ error estimation, we compare dynamic stress intensity factor obtained for $\omega = 0$, and the static stress intensity factor found from equlibrium equations.

For the solution of general eigenvalue problem, we used the simultaneous iteration algorithm.

The singularity of stress field in the vicinity of the crack was modeled by a special singular finite element in the form of a polygon with a cut and the displacement field in it was approximated from the solution for the cracked plane (Williams, 1957). Both for singular and regular elements the distributed mass matrix was used.

Results

The dynamic stress intensity factors were determined in a square plate with a central oblique crack under harmonic extension-compression. The crack angle was 45°, and a load of unit intensity was applied to the horizontal edges. The finite element mesh is shown on Fig. 2 (the square side a = 22 m, crack length $2\ell = 4\sqrt{2}$ m, Young's modulus $E = 1\,N/m^2$ the density $\rho = 0.1\ kg/m^3$, and the Poisson coefficient $\nu = 0.3$. Thirty modes were determined while solving the eigenvalue problem. The normalized static stress intensity factors are given by $K_{IS}/q\sqrt{\pi\ell} = 0.548$ and $K_{IIS}/q\sqrt{\pi\ell} = 0.626$. The value of K_{II} is written for right upper crack tip (it has the opposite sign at the left tip).

It was obtained that for stress intensity factor K_I the sum $\sum z_i = 1.017$, while $\sum z_i = 1.051$ for K_{II}. Hence, using the error estimation method introduced above, we can state that K_I and K_{II} are determined with an error 1.7% and 5.1% respectively. The plots of amplitudes of K_I (▲) and K_{II} (•) in the region $0 \leq \omega < \omega_1$ are shown in Fig. 2. It can be seen that the stress intensity factors are monotonically increasing functions from static values (when $\omega = 0$). These functions become unbounded when approaching the first natural frequency.

The evaluations with increasing number of modes involve large operational costs. Hence, in order to be sure that $z_i \to 0$ when $i > 30$, we used a coarsermesh and determined 60 modes. It was found that $z_i \approx 0$ when $i > 30$ and the modes after 30 have no influence on the stress intensity factor behaviour.

We have also considered a square plate with a central horizontal crack. The ratio of crack length to plate side was 0.364. The normalized stress intensity factor is equal to $K_{IS}/q\sqrt{\pi\ell} = 1.23$. In this problem we have found 16 modes. The first 3 modes correspond to rigid body motions with zero frequencies, while modes 4, 6, 7, 10–12, 14–16 correspond to zero values of the scalar product $(\{x^{(i)}\}^T\{f\}$. So, for a given load, only modes 5, 8, 9, 13 contribute to the sum (10), and the corresponding values z_i are equal to 0.920, -0.102, 0.212 and -0.014. The dimensionless frequencies for these modes are 4.985, 9.331,

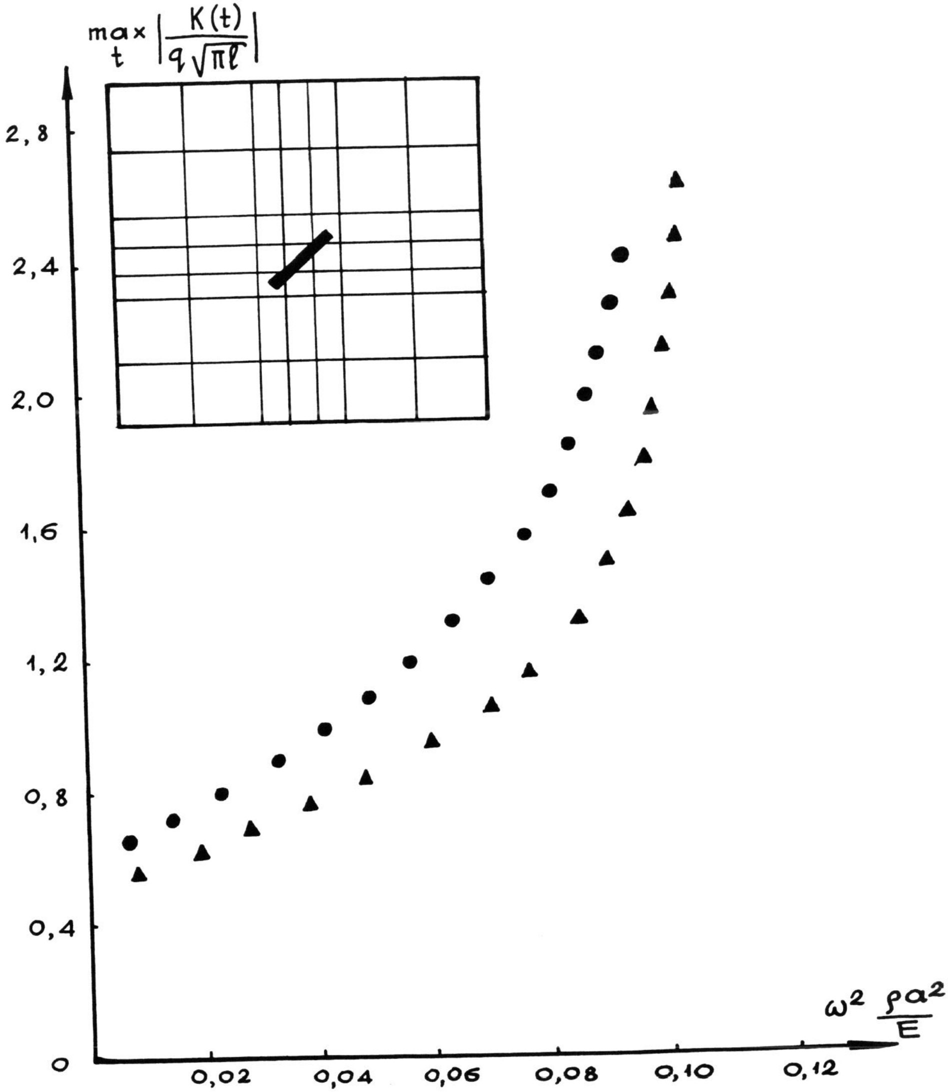

Fig. 2. The dependence of stress intensity factors on the frequency in the finite cracked plate.

11.084, 27.394. Hence, within the error 1%, we can write

$$K_I(t) = K_{IS}\left[\frac{0,920}{1-\left(\frac{\omega}{4,985}\right)^2} - \frac{0,102}{1-\left(\frac{\omega}{9,331}\right)^2} - \frac{0,212}{1-\left(\frac{\omega}{11,084}\right)^2} - \frac{0,014}{1-\left(\frac{\omega}{27,394}\right)^2}\right] e^{i\omega t} \quad (12)$$

REFERENCES

Boriskovsky, V.G. (1979). Stress intensity factor analysis for a vibrating cracked plate using the finite element method. J. Appl. Math. and Mech., 43, 764-768 (in Russian).

Boriskovsky, V.G., and V.Z.Parton (1983). Dynamic fracture mechanics. Advances in Science and Technology of VINITI (Itogi Nauki i Techniki VINITI). Ser. Mechanics of Deformable Solids, Vol. 16 (in Russian).

Parton, V.Z. (1972). The plane problem for the vibrations of a cracked strip. Proc. Math. Dept., MIChM, 45, 84-92 (in Russian).

Parton, V.Z., and E.M.Morozov (1978). Elastic-Plastic Fracture Mechanics. Mir Publ., Moscow.

Williams, M.L. (1957). On the stress distribution at the base of a stationary crack. J. Appl. Mech., 24, 109-114.