RECENT STUDIES OF ENERGY INTEGRALS AND THEIR APPLICATIONS

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ABSTRACT

In this paper, recent studies concerning path-independent integrals, of relevance in the fracture of solids, and the applications of these integrals, are critically summarized. Specific topics dealt with include: (i) unsteady dynamic crack-propagation in (nonlinear) elastic solids and (ii) slow stable, as well as fast crack propagation in elastic-plastic materials, which are characterized by an (incremental) flow theory of plasticity, and which are subject to arbitrary loading (and unloading) histories.

INTRODUCTION

It is almost redundant to say that parameters which quantify the severity of the crack-tip environment, but which may be evaluated as path-independent integrals based on far-field data, have played a dominant role in the enormous strides that have been made in the past 15 years or so in the subject of the mechanics of fracture. It is now well understood that the most widely used of such integrals, the so-called J integral, is valid theoretically, only in the context of incipient crack growth in (nonlinear) elastic materials. An excellent survey of the literature pertinent to the present topic of discussion has been earlier presented by Rice [1] in 1976 and by Bilby [2] at ICF4 in 1977. Since that time, a number of works dealing with theoretically valid 'path-independent' integral parameters in (nonlinear) elasto-dynamic crack propagation, and in quasi-static stable as well as fast fracture in elastic-plastic materials characterized by a flow theory of plasticity, have appeared. The present paper is an attempt at a summary of this literature, as well as that of some of the authors' work in this area.

Contents of this paper, in the order of their appearance, are: (i) a discussion of various path-independent integrals for dynamic crack propagation, in linear as well as nonlinear elastic materials, that have been introduced in literature, (ii) a critical evaluation of their validity as fracture parameters and their physical interpretation, if any, (iii) a discussion of general conservation laws in (nonlinear) elasto-dynamics and their relevance, or lack thereof, to the mechanics of fracture, (iv) certain path-independent integrals of relevance in slow stable as well as fast fracture, under arbitrary loading (and unloading) histories in materials characterized by (incremental) flow theory of plasticity, and (v) some illustrations of the applicability of path-independent integrals, which are shown to be relevant in this paper, in dynamic, elastic-plastic fracture problems.

ELASTO-DYNAMIC CRACK PROPAGATION

Preliminaries

We consider the material to be nonlinearly elastic and finitely deformed. We employ a fixed (global) cartesian coordinate system such that x_i and y_i refer, respectively, to the coordinates of a given material particle before and after deformation. We introduce another 'local' cartesian system X_i such that X_1 is locally normal to the crack border, X_2 normal to the crack plane, and X_3 is locally tangential to the crack border. The deformation gradient is represented by $F_{ij} = y_{i,j} \equiv (\partial y_i/\partial x_j)$ such that $dy_i = F_{ij}dx_j$. Henceforth in this section, we shall employ the 'nominal' stress, denoted here by t_{ij}, as the measure of stress in the deformed body. Note that $t_{ij} = (T_R)_{ji}$ where T_R is the first Piola-Kirchhoff stress [3].

The boundary-value problem in elasto-dynamics is in general posed by the equations [4],

(linear momentum balance): $$t_{ij,i} + f_j = \rho\ddot{u}_j \qquad (I.1)$$

(angular momentum balance): $$F_{ik}t_{kj} = F_{jk}t_{ki} \qquad (I.2)$$

(constitutive law): $$t_{ij} = \partial W/\partial F_{ji} \equiv \partial W/\partial e_{ji} \qquad (I.3)$$

$$\text{where, } e_{ji} = u_{j,i}\ ;\ u_j = y_j - x_j\ ;\ F_{ji} = e_{ji} + \delta_{ji}$$

(traction b.c): $$n_i t_{ij} = \bar{t}_j \quad \text{at} \quad S_t \qquad (I.4)$$

(displacement b.c): $$u_j = \bar{u}_j \quad \text{at} \quad S_u \qquad (I.5)$$

(initial conditions): $$u_j = \overset{o}{u}_j(x_k) \quad , \quad \dot{u}_j = \overset{o}{v}_j(x_k) \quad \text{at} \quad t = 0 \qquad (I.6)$$

In (I.1-6), all components refer to the fixed (global) cartesian system. In (I.1), f_j are body forces per unit initial volume, ρ is mass density of the undeformed body, and $\ddot{u}_j$ are accelerations; where $(\dot{\ })$ denotes a material derivative. Eq. (I.2) reduces to the condition of symmetry of the stress tensor if displacements and their gradients are infinitesimal. Eq. (I.3) is valid, in general, for an inhomogeneous as well as anisotropic body. The condition of material frame indifference imposes certain restrictions [3] on the structure of W; and hence, in general, it is a function only of $C_{ij} = F_{ki}F_{kj}$. When the structure of W is thus properly defined, condition

(I.2) becomes inherently embedded in the structure of W [see, for instance, Ref. 4]. In (I.4) and (I.5), S_t and S_u are parts of the external surface of the undeformed body, where tractions and displacements are respectively specified.

Self-Similar Crack Propagation

Consider the dynamic propagation of the crack in a self-similar fashion, such that the crack length increases by (da) in time (dt), with a non-constant velocity of propagation, c = (da/dt). The energy-release (or equivalently the work done in creating a new crack-surface) per unit of crack-extension, denoted by G, is given, from global energy-balance considerations, as:

$$G \equiv \int_{S_t} \bar{t}_i \frac{du_i}{da} ds - \frac{d}{da}\int_V (W + T)dv \qquad (I.7)$$

where $\bar{t}_i$ are external tractions on S_t, u_i are displacements, and W and T are, respectively, the strain and kinetic energy densities (per unit undeformed volume) of the cracked elastic body, V.

Now consider a "core" region, V_ε, near the crack-tip, which is enveloped by the contour Γ_ε. For instance, in two-dimensional problems Γ_ε may be considered to be a circle of radius ε, while in three-dimensional problems, it may be considered to be a toroidal surface whose axis of revolution coincides with the crack-front and whose cross-section is a circle of radius ε. Thus, the region $(V-V_\varepsilon)$ excludes the crack-tip. See Fig. 1 for further nomenclature. Considerations of energy balance in this region $(V-V_\varepsilon)$ leads to:

$$0 = \int_{S-\Gamma_\varepsilon} (t_i \frac{du_i}{da})ds - \frac{d}{da}\int_{V-V_\varepsilon} (W + T)dv \qquad (I.8)$$

It is clear that $S-\Gamma_\varepsilon$ is now the boundary of $(V-V_\varepsilon)$. Use of (I.8) in (I.7) results in the following relation for G:

$$G = \int_{\Gamma_\varepsilon} t_i \frac{du_i}{da} ds - \frac{d}{da}\int_{V_\varepsilon} (W + T)dv \qquad (I.9)$$

In self-similar crack propagation in an elastic material, wherein loading/unloading take place along the same path in a strain/stress space, it can be seen that the asymptotic (singular) solutions at the crack-tip at time t [when the crack-length is "a"] and at (t + dt) [when the crack-length is (a + da)] are self-similar. However, the strengths of such singular solutions may depend on the crack-lengths and hence may be different at time t and (t + dt). Based on these concepts, it can be shown [5] that:

$$\frac{d}{da}\int_{V_\varepsilon} (W + T)dv = \int_{\Gamma_\varepsilon} (W + T)N_1 ds - \int_{V_\varepsilon} \frac{\partial}{\partial a}(W + T)dv \qquad (I.10a)$$

and

$$\int_{\Gamma_\varepsilon} t_i \frac{du_i}{da} ds = \int_{\Gamma_\varepsilon} t_i (\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial X_1})ds \qquad (I.10b)$$

wherein, as seen from Fig. 1, X_1 is along the crack-axis, and N_1 is the direction cosine between the X_1 axis and a unit normal to the contour Γ_ε. Using (I.10) in (I.9), it is seen that:

$$G = \underset{\varepsilon\to o}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds - \int_{V_\varepsilon} [\frac{\partial}{\partial a} (W + T) - t_i \frac{\partial u_i}{\partial a}]dv \qquad (I.11)$$

It can be shown [5] that since $\partial(W + T)/\partial a$ has the same singular behaviour as $(W + T)$ itself, the second term in Eq. (I.11) tends to zero in the limit $\varepsilon\to o$; whereas the first term in (I.11) has a finite limit when $\varepsilon\to o$ [5]. Thus, we have the expression for energy-release rate:

$$G = \underset{\varepsilon\to o}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds \qquad (I.12)$$

as given in [5] and also in [6,7] though not as conclusively for a crack propagating with an arbitrary history of motion.

From a point of view of application in a numerical analysis of elasto-dynamic crack propagation in a finite body, it is preferrable to evaluate G from an integral over a far-field contour. To this end, we attempt to write:

$$G = \int_S [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds + R \qquad (I.13)$$

where S is the external surface. From (I.12) and (I.13), and realizing that the divergence theorem may be applied to terms $\partial W/\partial X_1$ and $\partial T/\partial X_1$ in the region $V-V_\varepsilon$, we see that:

$$R = \int_{\Gamma_\varepsilon} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds - \int_S [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds \qquad (I.14a)$$

$$= \int_{V-V_\varepsilon} [\rho(\ddot{u}_i - f_i) \frac{\partial u_i}{\partial X_1} - \rho\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1}]dv \qquad (I.14b)$$

In arriving at (I.14b), the conditions: (i) that in the considered elastic material, W is not an explicit function of X_1, i.e. the material is homogeneous along X_1 and (ii) that the dynamic equilibrium, (I.1) holds, have been used. Further, using a relation similar to that in (I.10a), we obtain:

$$\int_S t_i \frac{du_i}{da} ds - \frac{d}{da}\int_V (W + T)dv = \int_S t_i \frac{\partial u_i}{\partial a} ds - \int_V \frac{\partial}{\partial a} (W + T)dv$$
$$+ \int_S [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds \qquad (I.15)$$

Summarizing the relations (I.7,.9,.13,.14, and .15), we have:

$$G = \int_S t_i \frac{du_i}{da} ds - \frac{d}{da}\int_V (W + T)dv \qquad (I.16a)$$

$$= \int_S [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds + \int_S t_i \frac{\partial u_i}{\partial a} ds - \int_V \frac{\partial}{\partial a} (W + T)dv \qquad (I.16b)$$

$$= \int_S [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds + \underset{\varepsilon\to o}{\mathrm{Lt}} \int_{V-V_\varepsilon} [\rho(\ddot{u}_i - f_i) \frac{\partial u_i}{\partial X_1} - \rho\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1}]dv \qquad (I.16c)$$

$$= \int_{\Gamma+S_{c\Gamma}} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds + \underset{\varepsilon\to o}{\mathrm{Lt}} \int_{V_\Gamma-V_\varepsilon} [\rho(\ddot{u}_i - f_i)\frac{\partial u_i}{\partial X_1} - \rho\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1}]dv \qquad (I.16d)$$

$$= \underset{\varepsilon\to o}{\mathrm{Lt}} \int_{\Gamma_\varepsilon} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds \equiv J' \qquad (I.16e)$$

The sum of the far-field integral on Γ [which, it should be noted, is fixed in space] and the attendant volume integral, as in (I.16d) has been labeled J' in [8]. However, it was derived in [8], for infinitesimal deformations, based on a simple modification to a path-independent integral that was deduced from a general conservation law in elasto-dynamics in [5]. The sense of path-independence of J' embodied in (I.16) implies that for any closed volume V^* with a boundary Γ^* not enclosing the crack-tip, as in Fig. 1, we have:

$$\int_{\Gamma^*} [(W + T)N_1 - t_i \frac{\partial u_i}{\partial X_1}]ds + \int_{V^*} [\rho(\ddot{u}_i - f_i) \frac{\partial u_i}{\partial X_1} - \rho\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1}]dv = 0 \qquad (I.17)$$

which may easily be verified when (I.1) holds, and W is not an explicit function of X_1.

Because the use of J' as defined for any path Γ as in (I.16d), involves a volume-integral , the above notion of path-independence has been pronounced by many to be useless. The authors take an exception to this viewpoint, which they find to be somewhat orthodox. True, the evaluation of (I.16d) involves taking the limit of the volume integral to the crack-tip; and thus, on the surface, it appears to involve a "knowledge of the crack-tip fields", which the so-called J integral of elasto-statics [when $\dot{u}_i = \ddot{u}_i = 0$ in (I.16)] does not involve. First of all, it is clear from (I.16d) that its use does not require a knowledge of the crack-tip stress-strain fields, but only of displacement, velocity, and acceleration. Furthermore, a comparison of (I.16b) and (I.16c) reveals that

$$\underset{\varepsilon\to o}{\mathrm{Lt}} \int_{V-V_\varepsilon} [\rho\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1} + (f_i - \rho\ddot{u}_i) \frac{\partial u_i}{\partial X_1}]dV = \int_V \frac{\partial}{\partial a} (W + T)dV - \oint_{S_t} t_i \frac{\partial u_i}{\partial a} ds \qquad (I.18)$$

and thus, the l.h.s. of (I.18) remains finite in the limit $\varepsilon\to o$. This is interesting if one notes that, in known analytical asymptotic solutions [8], $\dot{u}_i \sim 0(r^{-1/2})$ and $\ddot{u}_i \sim 0(r^{-3/2})$; and hence, on first glance, the l.h.s. of (I.18) appears to contain non-integrable singularities. It has also been verified directly [8] that for known analytical asymptotic solutions for infinite bodies, the volume integral in (I.16) does have a finite limit, due to the fact that the angular variation of the integrand is such that:

$$\underset{\varepsilon\to o}{\mathrm{Lt}} \int_{-\pi}^{\pi} [\int_{o}^{\varepsilon} (\rho\ddot{u}_i u_{i,k}) r dr] d\theta \to 0 \qquad (I.19)$$

Even though finding the solution of u_i, $\dot{u}_i$, $\ddot{u}_i$ near the crack-tip in a finite body is a difficult problem analytically, it is a relatively simple task in computational mechanics. This has been demonstrated conclusively [9,10] by the authors in a variety of crack-propagation problems in finite bodies, even while using the simplest of crack-tip finite elements which do not model any of the singularities in strain, velocity, or acceleration.

If one considers the energy-release rate per unit time in self-similar elasto-dynamic crack propagation, one sees that this quantity is represented by:

$$CG = C\int_{\Gamma_\varepsilon} [(W + T)N_1 - \frac{\partial u_i}{\partial X_1}]dS \qquad (I.20a)$$

$$= \int_{\Gamma_\varepsilon} [(W + T)C_k n_k - C_k t_i \frac{\partial u_i}{\partial x_k}]ds \qquad (I.20b)$$

where C is the non-constant velocity of crack propagation along X_1 direction, N_1 is the component of a unit normal to Γ_ε along X_1, while C_k and n_k are components of the instantaneous velocity vector and the unit normal to Γ_ε, respectively, along x_k directions (see Fig. 1). [Note that the velocity vector $\underline{C}$ with $|C| = C$, along the X_1 direction in self-similar propagation, may be considered to have components C_k along x_k directions]. It is now a simple task (i) to apply the divergence theorem, (ii) use the coordinate invariant forms of the linear momentum balance laws of (I.1), under the assumption:

$$\frac{\partial W}{\partial x_k} = \frac{\partial W}{\partial u_{j,i}} \frac{\partial u_{j,i}}{\partial x_k} \qquad (I.21)$$

i.e., W does not depend explicitly on all the x_k (or the material is homogeneous in all the x_k directions), to derive from (I.20b):

$$CG = C_k J_k = \{\text{Lim}_{\varepsilon \to o} \int_{\Gamma + S_{c\Gamma}} [(W + T)n_k - t_i \frac{\partial u_i}{\partial x_k}]ds$$
$$- \int_{V_\Gamma - V_\varepsilon} [\rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial x_k} + (f_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial x_k}]dV\}C_k \qquad (I.22)$$

The sense of path-independence embodied in (I.22) is similar to that in (I.16,.17). In the above, $S_{c\Gamma}$ which is equal to $(S^+_{c\Gamma} + S^-_{c\Gamma})$ [+ and - referring, arbitrarily, to the crack faces] is the crack-surface enclosed within Γ, while S_c is the total crack-surface. Thus, an evaluation of J'_k not only involves a volume integral, but also an integral along the crack faces. The infinitesimal strain counterparts of the J'_k integrals have been first stated in [8], based on a simple modification to the J_k integrals for dynamic crack propagation given in [5].

It is important to note the meaning of (I.22)--it still governs the energy release per unit time, due to self-similar propagation (along X_1 axis). J'_k would simply characterize the total energy change due to a unit translation of the crack as a whole, rigidly, in the x_k direction. Thus, J'_k does not characterize the energy release due to a unit motion of the crack-tip in the x_k direction (and thus kinking the original crack). In fact there are no simple integrals that characterize the energy-release due to kinking of a crack, as is often erroneously implied in literature [15,16,29]. This is due to the fact that in deriving (I.10), which forms the basis of all the ensuing path-integrals thereof, use has been made of the self-similarity of solutions at time t and t + dt, which is valid only in self-similar crack propagation but not in the case, in general, of a kinked crack.

Assuming for the moment that the global and the crack-tip coordinates coincide, one may define:

$$J' = \underset{\varepsilon \to o}{Lt} \int_{\Gamma + S_{c\Gamma}} [(W + T)N_2 - t_i \frac{\partial u_i}{\partial X_2}]ds - \int_{V_\Gamma - V_\varepsilon} [\rho \dot{u}_i \frac{\partial \dot{u}_i}{\partial X_2} + (f_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial X_2}]dV \qquad (I.23)$$

which would characterize the total energy change for a unit rigid translation of the crack as a whole (and not a unit growth of the crack-tip alone) in the X_2 direction. Assuming zero body force, traction-free

crack-faces, and elasto-static deformations, one may reduce (I.23) to:

$$J_2 = \int_\Gamma (WN_2 - t_i \frac{\partial u_i}{\partial X_2})ds + \underset{\varepsilon \to o}{Lt} \int_{S^+_{c\Gamma}} (W^+ - W^-)dS \qquad (I.24)$$

wherein, for a flat crack-face, $N^+_2 = - N^-_2 = - 1$. The definition of J_2 of Budiansky and Rice [11], on the other hand, does not involve the crack-face integral, which accounts for discontinuities of W along the crack-face. Thus, as also noted by [2,12], J_2 as given by [11] is not path-independent. Even though (I.24) appears to involve a knowledge of crack-tip W for its successful application as a path-independent integral, the use of (I.24) has been conclusively demonstrated [10,13] in computational approaches using simple (non-singular) crack-tip finite elements.

From the above discussions, it should be clear that neither the integrals J'_k nor any other similarly "path-independent" integrals provide any information as to kinking of a crack, or of the direction of propagation of the crack-tip in anything other than a collinear fashion, contrary to speculations often made in literature [14,15,16].

Using the asympotic solutions in self-similar crack propagation, even under arbitrary time history of motion of the crack-tip, viz., $\dot{u}_i \sim - C\partial u_i/\partial X_i$, it is seen that the energy-release rate expression in (I.20a) reduces to that of Freund [17]. It is worth noting that (I.20a) as well as Freund's result are valid for an arbitrary shape of the loop Γ_ε near the crack-tip. On the other hand, if consideration is restricted to steady-state (i.e., the field is invariant w.r.t. an observer moving with the crack-tip) self-similar propagation at a constant crack-tip velocity, it is seen that everywhere in V, one has: $\dot{u}_i = - C\, \partial u_i/\partial X_1$; $\dot{u}_{i,1} = - C\partial^2 u_i/\partial X_1^2$; $\ddot{u}_i = C^2\partial^2 u_i/\partial X_1^2$. [Note, however, even at constant velocity, unsteady conditions in general imply that: $\dot{u}_i = (\partial u/\partial t) - C_1(\partial u_i/\partial X_1)$, and $\ddot{u}_i = (\partial^2 u_i/\partial t^2) + C^2(\partial^2 u_i/\partial X_1^2) - 2C\, (\partial^2 u_i / \partial X_1\, \partial t)$]. Thus, when body forces $f_i = 0$, for steady-state, constant velocity propagation, the volume integral in (I.16d) disappears; and the resulting expression, with $2T = \rho C^2(\partial u_i/\partial X_1)^2$, becomes identical to that given by Sih [18], even though the far-field contour considered in [18] moves along with the crack-tip at the same velocity. It may be noted, however, that such steady-state constant velocity propagation seldom occurs in practical problems of fast fracture in finite bodies; see [19] for further details.

In as much as $J'(\equiv G)$ as defined in (I.16e) has a well-defined physical meaning as the crack-tip energy release rate and can be conveniently computed from simple numerical procedures from far-field quantities through (I.16d), it can be used as a parameter governing elastodynamic crack propagation and arrest. The relation between J'_k and the dynamic stress-intensity factors are given in [8]. J' is in general a function of the crack-tip velocity [8]. In a dynamic fracture problem, initiation of propagation occurs at $J' = J'_d$ and during crack propagation, $J' = J'_D(C)$ where J'_d and J'_D are material properties. Examples of prediction of crack-propagation histories and crack-arrest using these criteria, and comparison with experimental results may be found in [9,10,13,19,20].

As noted, the far-field path Γ in (I.16d) is fixed in space. On the other hand, considering a far-field contour Γ to be a rigid path surrounding the crack-tip and in translation at the same velocity C along the X_1 axis, a path-independent integral, denoted here by J*, was given by Bui [21,22] and Erlacher [23] for infinitesimal deformation:

$$J^* = \int_\Gamma [Wn_1 - n_k\sigma_{kj}u_{j,i} - Tn_1 - \rho\dot{u}_j u_{j,1}C_1 n_1]ds + \frac{D}{Dt}\int_V \rho\dot{u}_j u_{j,1} dV \quad (I.25)$$

When the material derivative for a moving control volume containing singularities is properly treated, it may be shown [see 8,19] that (I.25) is equivalent to (I.16d). It is, however, the experience of the authors that (I.16d) with a fixed path is easier to use directly in a computational scheme [9,10,13,19,20].

For linear elastic materials undergoing infinitesimal deformations, Irwin [24] and Erdogan [25] gave the expression for energy-release rate in dynamic crack propagation, as:

$$CG(t_o) = -\tfrac{1}{2}\left(\frac{d}{dt}\right)_{t_o}\int_{a(t_o)}^{a(t)} t_i(X_1,t_o)u_i[(X_1 - \langle a(t) - a(t_o)\rangle),t_o]dX_1 \quad (I.26)$$

Thus, it is the work of tractions at t_o in moving through the displacements at the corresponding points at time $t_o + dt$. The validity of (I.26) has been established for linear elasto-dynamics by Gurtin and Yatomi [26]. On the other hand, Achenbach [27] gives, for finite deformations as well as nonlinear elastic behaviour, the expression for G, as:

$$G = \left(\frac{1}{C}\right)\lim_{\varepsilon\to o}\int_{a-\varepsilon}^{a+\varepsilon}\sigma_{i2}(X_1,0,t)[\dot{u}_i(X_1 0^+,t) - \dot{u}_i(X_1,0^-,t)]dX_1 \quad (I.27)$$

where $\dot{u}_i(X_1,0^\pm,t)$ denote the particle velocities at the crack surfaces, $X_2 = 0^\pm$. The tip of the crack is denoted by $X_1 = a$ and ε a small number. Now, consider the path Γ_ε in (I.12) to be a rectangle of height 2δ (in X_2 direction) and width 2ε (in the X_1 direction) and centered at the crack-tip. Thus, we may write from (I.12) that:

$$G = \underset{\varepsilon\to o}{Lt}\ \underset{\delta\to o}{Lt}\int [(W + T)N_1 - t_i\frac{\partial u_i}{\partial X_1}]ds \quad (I.28)$$

$$= \underset{\varepsilon\to o}{Lt}\ \underset{\delta\to o}{Lt} - \int t_i\frac{\partial u_i}{\partial X_1}ds \quad (I.29)$$

since N_1 is zero on segments parallel to X_1 axis, and the integral of $(W + T)N_1$ vanishes along segments parallel to X_2 axis in the limit $\delta\to o$. Also, near the crack-tip, $\dot{u}_i \simeq - C\,\partial u_i/\partial X_1$. Thus, it appears on first glance that (I.27) is the correct limit of (I.29). However, this has conclusively been disproved by Yatomi [28] who shows that the limit $\delta\to o$ must be taken after the integral in (I.29) is evaluated; and in any event, (I.12) always leads to the correct result, even for finite deformation nonlinear elastic problems, irrespective of the shape of Γ_ε. For the special path described above, the general validity of (I.28,.29) is established by Gurtin and Yatomi [26].

Strifors [15] and Carlsson [16], on the other hand, apply the "principle of virtual work" to an arbitrary part V_Γ of the body containing a crack (as in Fig. 1), which they consider to have a "finite cohesive zone". Their [15,16] definition of "an apparent crack-extension force", written below, for instance, in the X_1 direction, is arrived at by them [15,16] by considering virtual displacements of the form $\delta u_i = -\partial u_i/\partial X_1$ over V_Γ as well as over a cohesive zone of size ε, as:

$$F = \int_{V_\Gamma}[t_{ij}(\partial u_{j,i}/\partial X_1) - (f_i - \rho\ddot{u}_i)(\partial u_i/\partial X_1)]ds$$

$$-\int_{\Gamma+S_{c\Gamma}} t_i(\partial u_i/\partial X_1)ds \equiv \underset{\varepsilon\to o}{Lt}\int_o^\varepsilon t_i^+[(\partial u_i/\partial X_1)^- - (\partial u_i/\partial X_1)^+]dX_1 \quad (I.30)$$

From the preceding arguments it may be seen that the extreme right-hand side of Eq. (I.30) does not have the meaning of an energy-release rate even in the limited situations when (I.27) may be valid, because of the only one-sided limit of integration appearing in (I.30). Further, since $t_{ij}(\partial u_{j,i}/\partial X_1)$ and $(\ddot{u}_i\partial u_i/\partial X_1)$ may have singularities of order greater than (r^{-1}), the limit of the integral over V_Γ must be considered separately. Thus, even though F in (I.30) is path independent, its meaning is not clear. Kishimoto, Aoki, and Sakata [14] define a parameter such that:

$$\hat{J} = -\int_{\Gamma_\varepsilon} t_i(\partial u_i/\partial X_1)dS \quad (I.31)$$

where Γ_ε is a non-distorting 'small' contour which moves at the same speed as the crack-tip. Even though $\hat{J}$ as in (I.31) is defined in [14] as one of the components needed in analyzing crack-growth at an angle to the initial direction, this concept is questionable for reasons discussed earlier. Further, for arbitrary Γ_ε, $\hat{J}$ as in (I.31) is the rate of work done on the process zone of size Γ_ε, by the surrounding medium and is not the energy release to the crack-tip. From (I.31), and the divergence theorem, they [14] derive the "far-field" expression:

$$\hat{J} = \int_{\Gamma+S_{c\Gamma}}(WN_1 - t_i\frac{\partial u_i}{\partial X_1})dS + \int_{V-V_\varepsilon}(\rho\ddot{u}_i - f_i)\frac{\partial u_i}{\partial X_1}dV - \int_{\Gamma_\varepsilon} WN_1 ds \quad (I.32)$$

Note the presence of a near-field integral on Γ_ε, in the "far-field" expression (I.32). In [14], this integral over Γ_ε on the r.h.s. of (I.32) is dropped (see Eqs. 24,25 of [14]), by considering a special case of Γ_ε to be a rectangle of size (2ε x 2δ) centered at the crack-tip. However, it should be noted that the integral of WN_1 over Γ_ε does not vanish for arbitrary Γ_ε. Also if the integral over Γ_ε is dropped from (I.32), and the resulting integral is considered in the limit when Γ is shrunk to Γ_ε, one obtains a near-field definition of $\hat{J}$ from (I.32) that is different from the original definition, (I.31)!. Kishimoto et al. [29] in a later paper, redefine $\hat{J}$ as

$$\hat{J} = \int_{\Gamma_\varepsilon}[WN_1 - t_i(\partial u_i/\partial X_1)]dS \quad (I.33)$$

$$= \int_{\Gamma+S_{c\Gamma}}[WN_1 - t_i(\partial u_i/\partial X_1)]dS + \int_{V-V_\varepsilon}(\rho\ddot{u}_i - f_i)(\partial u_i/\partial X_1)dV \quad (I.34)$$

and consider [29] $\hat{J}$ in (I.33,.34) as the "energy release rate per unit of crack translation in the X_1 direction". Comparing (I.33) with (I.12), it is seen that such is not the case for arbitrary shapes of Γ_ε, since (I.33) does not contain the rate of change of kinetic energy in the energy-balance for dynamic crack-growth.

It is easy to see that:

$$\int_{V-V_\varepsilon}\rho\ddot{u}_i(\partial u_i/\partial X_1)dV = \int_{V-V_\varepsilon}[\partial(\rho\ddot{u}_i u_i)/\partial X_1 - \rho u_i(\partial\ddot{u}_i/\partial X_1)]dV$$

$$= \int_{\Gamma+S_{c\Gamma}}(\rho\ddot{u}_i u_i N_1 ds)ds - \int_{\Gamma_\varepsilon}\rho\ddot{u}_i u_i N_1 ds - \int_{V-V_\varepsilon}\rho u_i(\partial\ddot{u}_i/\partial X_1)dV \quad (I.35)$$

Using (I.35) and the rather extraordinary case when f_i are constants [i.e., $f_i \neq f_i(x_k)$], one may derive from (I.34), what Ouyang [30] defines as a parameter Y_3 for elasto-dynamic crack propagation (and an associated parameter Y_6, slightly different from Y_3, to account for plasticity), as:

$$Y_3 = \hat{J} + \int_{\Gamma_\varepsilon} \rho \ddot{u}_i u_i N_1 ds \equiv \int_{\Gamma_\varepsilon} [(W + \rho \ddot{u}_i u_i) N_1 - t_i(\partial u_i/\partial X_1)] ds$$

$$= \int_{\Gamma+\Gamma_s} [WN_1 - t_i(\partial u_i/\partial X_1) + (\rho \ddot{u}_i - f_i) u_i N_1] ds - \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{V-V_\varepsilon} \rho u_i (\partial \ddot{u}_i/\partial X_1) dV \tag{I.36}$$

Comparing (I.36) with (I.12) it is seen that Y_3 is not in general an energy-release rate for elasto-dynamic crack-propagation, and hence its use as a fracture parameter is questionable. Likewise, the parameter Y_2 of [30], which is the integral in time of Y_2, (similar to the time integral of G of (I.12) which would give the total fracture energy up to the current time t) is not a meaningful fracture parameter.

More recently, however, Aoki, Kishimoto and Sakata [31] define a parameter which, for elasto-dynamic crack-propagation, may be defined as:

$$\hat{J} = \int_{\Gamma end} \{(W + T)N_1 - t_i(\partial u_i/\partial X_1)\} ds \tag{I.37}$$

$$\equiv \int_{\Gamma+\Gamma_s} [WN_1 - t_i(\partial u_i/\partial X_1)] ds + \int_{V-V_\varepsilon} (\rho \ddot{u}_i - f_i)(\partial u_i/\partial X_1) dV + \int_{\Gamma_\varepsilon} TN_1 ds \tag{I.38}$$

Note the presence of a "near-tip" integral (over Γ_ε) in the supposedly far-field expression (I.38) for $\hat{J}$. They [31] go on to consider the limit of (I.38) for two different shapes of Γ_ε. In any event, the presence of the integral over Γ_ε makes it inconvenient to use (I.38) in a meaningful computational sense (i.e., without an accurate near-tip modelling). It is a simple matter to use the divergence theorem and eliminate the integral over Γ_ε from (I.38); in which case, the resulting far-field expression for the energy release rate is none other than J', of Eq. (I.16d).

Finally we mention the following path-independent integrals of Nilsson [32] and Gurtin [33], respectively, for a stationary crack in a linear elasto-dynamic field:

$$I(p) = \int_C [(\bar{W} + \tfrac{1}{2}\rho_o p^2 \bar{u}_i \bar{u}_i) N_1 - \bar{t}_i(\partial \bar{u}_i/\partial X_1)] ds \tag{I.39}$$

and

$$I = \tfrac{1}{2} \int_\Gamma [(\sigma_{jk} * u_{j,k} + \rho u_j * \ddot{u}_j) N_1 - N_k \sigma_{jk} * \partial u_j/\partial X_1)] ds \tag{I.40}$$

In (I.39) I(p) denotes a Laplace transform of I(t), and ($^-$) denotes a Laplace transform of ( ). Likewise, in (I.40), (f) * (g) denotes a convolution integral in the time domain, of two functions f(t) and g(t). Thus, both (I.39) and (I.40) do not easily give the instantaneous value of the crack-tip parameter, which is useful in analyzing dynamic crack propagation and arrest in a finite body. Further, in the case of a stationary crack in a dynamic field, the energy release due to incipient crack growth at any instant of time is given from (I.12) as:

$$G_{stationary} = \int_{\Gamma_\varepsilon} [WN_1 - t_i(\partial u_i/\partial X_1)] ds \tag{I.41}$$

$$= \int_\Gamma [WN_1 - t_i(\partial u_i/\partial X_1)] ds - \underset{\varepsilon \to 0}{\mathrm{Lt}} \int_{V_\Gamma - V_\varepsilon} (f_i - \rho \ddot{u}_i) \frac{\partial u_i}{\partial X_1} dV \tag{I.42}$$

(I.41) follows from (I.12) since T is no longer singular at the stationary crack-tip; (I.42) follows from (I.41) due to the divergence theorem.

We now turn to a class of "path-independent integrals" derivable from the application of Noether's theorem [34] in the form of "conservation laws" [7,35-38].

"Conservation Laws" and Their Relevance to Fracture Mechanics

The density of the "Lagrangean" for a (linear or nonlinear) elasto-dynamic problem is defined as L = (W-T-P) where W is the strain energy density, T the kinetic energy density, and P the potential of external forces. In Lagrange's description of motion (with material coordinates x_i as independent variables), L may be considered, in general, to be a function of the variables $y_{i,j} (= \partial y_i/\partial x_j)$ (or equivalently of $u_{i,j}$), $\dot{u}_i$, u_i, as well as that of the independent variables x_i (for a non-homogeneous system) and t (for a nonholonomic system). Thus,

$$L^* = \int_{t_o}^{t} \int_V L(x_i, u_i, \dot{u}_i, u_{i,j}, t) dv dt \tag{I.43}$$

Noether's theorem [34] concerning the invariance of L* w.r.t. certain transformations of the arguments of L leads to corresponding conditions which may be labelled as conservation laws. Eshelby [7,39,40] was the first to intuitively recognize the importance of these in connection with 'forces' on point defects and cracks. Gunther [35] was apparently the first to apply the formalism of the Noether's theorem, to obtain general conservation laws in elastostatics. Knowles and Sternberg [36] provided, independently, a thorough treatment also in the case of finite elastostatics; and this work was later extended by Fletcher [37] to linear elasto-dynamics, although the claim in [37] that equations (3.1-.4,.6) therein can easily be extended to finite elasticity should be viewed with some caution. More recently Golebiewska-Herrmann [38,41] has embarked on a study of conservation laws in finite elasto-dynamics using both Lagrange as well as Eulerean descriptions of motion, although the attempt in [41] to relate these only to elastostatic fracture mechanics appears formal at best.

Here we briefly discuss the case of Lagrange's description of motion and consider the conservation laws that arise from (I.43) when it is required to be invariant under various transformations, when body forces f_i are present:

(i) invariance under time translation ($\tilde{t} = t + \varepsilon$):

$$-\frac{d}{dt}(W + T) + f_i \dot{u}_i + \frac{\partial}{\partial x_k}(t_{kj}\dot{u}_j) = -\frac{\partial L}{\partial t}\Big|_{explicit} \tag{I.44a}$$

when L does not depend on t explicitly, this leads to a 'conservation law' for a closed volume V* (with a surface *, see Fig. 1) that does not contain the crack-tip:

$$\frac{d}{dt}\int_{V^*}(W + T)dv - \int_{V^*} f_i\dot{u}_i dv - \int_{\Gamma^*} n_k t_{ki}\dot{u}_i ds = 0 \qquad \text{(I.44b)}$$

Eq. (I.44b) is analogous to the energy-balance relation (I.8) except for a subtle difference: (I.44b) does not imply any crack-growth, whereas (I.8) is written specifically for crack-growth.

(ii) invariance under translation of $y_i(\tilde{y}_i = y_i + \varepsilon_i)$:

$$t_{ij,i} + f_j = \rho\ddot{u}_j \qquad \text{(I.45a)}$$

or

$$\int_{\Gamma^*} n_i t_{ij} ds + \int_{V^*} f_j dv - \int_{V^*} \rho\ddot{u}_j dv = 0 \qquad \text{(I.45b)}$$

which are, respectively, local and global equations of balance of linear momentum.

(iii) invariance under rotations of $y_i(\tilde{y}_i = y_i + e_{ijk}\omega_j y_k)$:

Here, e_{ijk} is the alternating tensor. The balance law is:

$$-\frac{d}{dt}(e_{ijk}\rho\dot{u}_k y_j) + \frac{\partial}{\partial x_p}(e_{ijk}t_{pk}y_j) + e_{ijk}f_k y_j = 0 \qquad \text{(I.46a)}$$

When (I.45a) is used, it is seen that (I.46a) is but a disguised form of the angular momentum balance, (I.2). The corresponding 'conservation law' is:

$$\int_{V^*} e_{ijk}y_j(f_k - \rho\ddot{u}_k)dv + \int_{\Gamma^*} e_{ijk}y_j n_p t_{pk} ds = 0 \qquad \text{(I.46b)}$$

Note that V^* is the volume in undeformed configuration.

(iv) invariance under translation of $x_i(\tilde{x}_i = x_i + \varepsilon_i)$:

Note that translation of the coordinates in the undeformed body are considered (or equivalently the translation of the elastic field referred to the undeformed geometry). The balance law is:

$$\frac{d}{dt}(\rho\dot{u}_j u_{j,i}) - f_k u_{k,i} + \frac{\partial}{\partial x_k}[(W - T)\delta_{ik} - t_{kj}u_{j,i}] = \left(\frac{-\partial L}{\partial x_i}\right)_{exp.} \qquad \text{(I.47a)}$$

When we consider: (i) a volume V^* that does not contain any singularities such that each of the terms in (I.47a) is integrable in V^*, and (ii) L does not explicitly depend on x_i, i.e., the material is homogeneous in all the x_i directions, we obtain from (I.47a) the conservation law:

$$\int_{V^*} \left\langle \frac{d}{dt}(\rho\dot{u}_j u_{j,i}) - f_k u_{k,i} \right\rangle dv + \int_{\Gamma^*} [(W - T)n_i - t_j u_{j,i}]ds = 0 \qquad \text{(I.47b)}$$

The above conservation law, and it alternate representations, were discussed in [5]. Note that Eshelby [7,39,40] names the terms in brackets [ ] in (I.47) as the "energy-momentum tensor".

(v) invariance under rotations of x_i:

This is possible only when the (linear or nonlinear elastic) material is isotropic. The balance law is:

$$0 = e_{ijk}x_k[(\rho\ddot{u}_m - f_m)y_{m,i} + \rho u_m u_{m,i}] + \frac{\partial}{\partial x_p}\langle e_{ijk}x_k[(W - T)\delta_{ip} - t_{pm}y_{m,i}]\rangle \qquad \text{(I.48a)}$$

When the global angular momentum conservation law (I.46a) is used, the conservation law corresponding to (I.48a) can be written as:

$$0 = e_{ijk}\left\{\int_{V^*}[(\rho\ddot{u}_m - f_m)u_{m,i}x_k + \rho\dot{u}_m\dot{u}_{m,i}x_k - \rho\ddot{u}_i u_k]dv + \int_{\Gamma^*}[(W - T)x_k n_i + n_m t_{mi}u_k - n_p t_{pm}u_{m,i}x_k]ds\right\} \qquad \text{(I.48b)}$$

where V^* is the volume that is void of any singularities.

(vi) invariance under scale changes of $x_i, t[\tilde{x}_i = (1 + \varepsilon)x_i,\ \tilde{t} = (1 + \varepsilon)t]$:

This is possible only when the material is linear. The corresponding conservation law in linear elasto-dynamics is [37]:

$$\int_{V^*}\frac{d}{dt}(\rho e_{ijk}u_k\dot{u}_j + \rho e_{ijk}x_j\dot{u}_m u_{m,k})dv + \int_{\Gamma^*}(e_{imj}u_m\sigma_{jk}n_k - e_{imj}x_j u_{l,m}\sigma_{lk}n_k + e_{imk}n_k x_m L)ds = 0 \qquad \text{(I.49)}$$

Now we consider the application of the conservation laws (I.47b) and (I.48b) to nonlinear elasto-dynamic crack propagation. We consider a volume $V_\Gamma - V_\varepsilon$ which does not contain the crack-tip, where Γ is any path enclosing the crack-tip, V_ε is a small volume with the boundary Γ_ε also enclosing the crack-tip; thus $\Gamma + S_{c\Gamma} - \Gamma_\varepsilon$ is the boundary of $V - V_\varepsilon$. Note that the divergence theorem, in the presence of possible non-integrable singularities, may be applied only in $V - V_\varepsilon$ in the limit as $\varepsilon \to 0$. Based on these arguments, further elaborated upon in [5], we obtain from (I.47b) the path-independent integrals:

$$J_k \equiv \underset{\varepsilon\to 0}{\mathrm{Lt}} \int_{\Gamma_\varepsilon}[(W - T)n_k - t_j u_{j,k}]ds \qquad \text{(I.50a)}$$

$$= \underset{\varepsilon\to 0}{\mathrm{Lt}} \int_{\Gamma + S_{c\Gamma}}[(W - T)n_k - t_j u_{j,k}]ds + \int_{V_\Gamma - V_\varepsilon}\left\langle \frac{d}{dt}(\rho\dot{u}_j u_{j,k}) - f_m u_{m,k}\right\rangle dv \qquad \text{(I.50b)}$$

Comparing (I.50a,b) with (I.20b and .22), it should be evident that J_k of (I.50a) are not associated with the concept of an energy-release rate, but rather (as the roots of their derivation would indicate) are associated with the rate of change of Lagrangean L^* of the system, due to unit translation of the crack in the x_k direction [5]. That the equivalent "energy-momentum tensor" in elasto-dynamics does not lead to an energy-release rate was also noted by Eshelby [7]. Thus, the relevance of (I.50a) as a 'fracture parameter' is vaccuous. Likewise, the integral of (I.50b) in time, say,

$$Y_1 = \int_{t_0}^{t} J_1 dt = \underset{\varepsilon\to 0}{\mathrm{Lt}} \int_{t_0}^{t}\left\langle \int_{\Gamma + S_{c\Gamma}}[(W - T)n_1 - t_j u_{j,1}]ds\right\rangle dt$$

$$- \int_{t_o}^{t} < \int_{V_\Gamma - V_\varepsilon} (f_k u_{k,1}) dv > dt + \int_{V_\Gamma - V_\varepsilon} \rho \dot{u}_j u_{j,1} dv \Big|_{t_o}^{t} \qquad \text{(I.51)}$$

has little relevance to fracture--it is the total change of Lagrangean from t_o to t. Eq. (I.51), in a slightly less general form, for the case of infinitesimal deformation, along with the assumption of a rather special set of <u>constant</u> body forces, i.e., $f_k \neq f_k(x_i)$ [which renders the volume integral of $f_k u_{k,1}$ to be a surface integral of $f_k u_k n_1$], appears in a paper by Ouyang [30].

If one realizes the identity:

$$\int_{\Gamma_\varepsilon} 2Tn_k dS = \int_{\Gamma + S_{c\Gamma}} 2Tn_k ds - \int_{V_\Gamma - V_\varepsilon} 2\rho \dot{u}_m \dot{u}_{m,k} dv \qquad \text{(I.52)}$$

and adds (I.52) to (I.50), one recovers the integrals J_k' of (I.20b and .22) which are associated with the energy release rate, as was done originally in [8].

Analogous to the way in which (I.50) is derived from (I.47b), we may derive the following path-independent integrals from (I.48b):

$$\begin{aligned} L_j &= e_{ijk} \int_{\Gamma_\varepsilon} [(W - T)x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,i} x_k] ds \\ &\equiv e_{ijk} \{ \int_{\Gamma + S_{c\Gamma}} [(W - T)x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,i} x_k] ds \\ &\quad + \int_{V_\Gamma - V_\varepsilon} [(\rho \ddot{u}_m - f_m) u_{m,i} x_k + \rho \dot{u}_m \dot{u}_{m,i} x_k - \rho \ddot{u}_i u_k] dv \} \end{aligned} \qquad \text{(I.53)}$$

which would have the meaning of the rate of change of Lagrangean L per unit rotation of the crack. In order to obtain an equivalent "energy-release" interpretation, we may add the identity

$$\int_{V_\Gamma - V_\varepsilon} \frac{\partial}{\partial x_i} (2Te_{ijk} x_k) dv - \int_{\Gamma + \Gamma_s} 2e_{ijk} T x_k n_i ds = \int_{\Gamma_\varepsilon} 2e_{ijk} T x_k n_i ds \qquad \text{(I.54)}$$

to (I.53) and obtain:

$$\begin{aligned} L_j' &= e_{ijk} \int_{\Gamma_\varepsilon} [(W + T)x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,i} x_k] ds \\ &= e_{ijk} \{ \int_{\Gamma + S_{c\Gamma}} [(W + T)x_k n_i + n_m t_{mi} u_k - n_p t_{pm} u_{m,i} x_k] ds \\ &\quad \int_{V - V_\varepsilon} [(\rho \ddot{u}_i - f_m) u_{m,i} x_k - \rho \dot{u}_m \dot{u}_{m,i} x_k - \rho \ddot{u}_i u_k] dv \} \end{aligned} \qquad \text{(I.55)}$$

Finally, we note that the so-called M-integral for <u>linear</u> elasto-dynamics can be derived from (I.49).

<u>Complementary Representation of Path-Independent Integrals in (Nonlinear) Elasto-Dynamic Fracture</u>

Here we define the complementary energy density (per unit initial volume) of the material, denoted here by W., through the contact (Legendre) transformation,

$$W_c(t_{ij}) = t_{ij} u_{j,i} - W(u_{j,i}) \qquad \text{(I.56)}$$

Evaluation of (I.56), however, involves finding the inverse of the stress-strain relation,

$$t_{ij} = \partial W / \partial u_{j,i} \qquad \text{(I.57)}$$

That the inverse of (I.57) is not unique is now well established [3, 4]. However, by defining the so-called 2nd Piola-Kirchhoff stress tensor,

$$S_{ij} = t_{ik} (\partial x_j / \partial y_k) \qquad \text{(I.58)}$$

we may define a 'valid' complementary energy:

$$W_c(S_{ij}) = S_{ij} C_{ij} - W(C_{ij}) \qquad \text{(I.59)}$$

where $C_{ij} = (y_{k,i} y_{k,j})$

Evaluation of (I.59) involves finding the inverse of

$$S_{ij} = (\partial W / \partial C_{ij}) \qquad \text{(I.60)}$$

which is known to be unique [3,4], such that:

$$C_{ij} = \partial W_c / \partial S_{ij} \qquad \text{(I.61)}$$

We now define a Lagrangean in terms of the complementary energy, W_c, as:

$$L(S_{ij}, u_i, \dot{u}_i, u_{i,j}, x_i, t) = \int_{t_o}^{t} \int_{V^*} [S_{ij} C_{ij} - W_c(S_{ij}) - T - P] dv dt \qquad \text{(I.62)}$$

By applying Noether's theorem, it is now possible to derive a variety of conservation laws, and complementary path-independent integrals, from (I.62). We omit further details, but refer for examples of these to [42,43].

We conclude this section by noting that of the seemingly infinite varieties of "path-independent integrals" and attendant "conservation laws" possible in nonlinear (or linear) elasto-dynamic crack propagation, only J' of (I.16) and (I.22), and the equivalent J* of (I.25) have the property: (i) they characterize an energy release rate due to crack propagation, (ii) they are measurable, and (iii) they are measures of dynamic crack-tip fields. In linear elasto-dynamic crack propagation, even though some of the other "path-independent integrals", such as $\hat{J}$ of (I.31) and (I.33), and J_k of (I.50a), do not have the same physical meaning as (J' and J*), they may be related to the dynamic stress-intensity factors [k(t)]. Such relations, which are of course different from those between J' and k(t), are given in [8].

INELASTIC (AND DYNAMIC) CRACK PROPAGATION

We first consider crack-growth initiation and stable growth, under quasi-static conditions, in elastic-plastic materials. The most widely used parameter so far, and the one that has made possible certain impressive advances in elasto-plastic fracture has been the J integral. In the context of incipient self-similar growth, under quasi-static conditions, of a crack in an elastic material, J [which is equal to J' when $\dot{u}_i$ and $\ddot{u}_i$ are set to zero in (I.16)] has the meaning of energy-release per unit of crack extension. As in the case of J' of (I.16), the path-independency of J, evaluated now only as a contour integral, can be established when the strain energy density of the material is a single-valued function of strain and the material is appropriately homogeneous. In a deformation theory of plasticity, which is valid for radial monotonic loading but precludes unloading (and thus is essentially and mathematically equivalent to a nonlinear theory of elasticity), J still characterizes the crack-tip fields. However, in this case J does not have the meaning of an energy-release rate; it is simply the total potential-energy difference between identical and identically (monotonically) loaded cracked bodies which differ in crack lengths by a differential amount. It should be emphasized that even this interpretation of J under a deformation theory of plasticity is valid only up to the point of crack growth initiation [43]. Moreover, in a flow theory of plasticity, under arbitrary load histories, the path-independence of J, evaluated as a contour integral, is no longer valid; and further, under these circumstances, J does not have any physical meaning.

However, significant advances have been made, in the past decade, in the problem of crack-growth initiation in monotonically loaded structures, using the concept of J integral. The principal contributions that made these possible may perhaps be identified, as: (i) the work of Hutchinson [44] and Rice and Rosengren [45], who show that the stresses and strains near the crack-tip in a monotonically loaded body of a power-law hardening material, under yielding conditions varying from small-scale to fully plastic, are controlled by J; (ii) the work of Begley and Landes [46] and Rice et al. [47] on the measurement of J from small laboratory test specimens; and (iii) simple procedures for estimation of J, by interpolating between fully-plastic solutions and elastic solutions, based on the works of Bucci et al. [48], Shih and Hutchinson [49], and Rice et al. [47]. On the other hand, a large amount of crack growth in a ductile material is necessarily accompanied by a significant non-proportional plastic deformation which invalidates the deformation theory of plasticity. Thus, the validity of J, as a contour integral defined by Eshelby [40] and Rice [50], is questionable under these circumstances. For limited amounts of crack-growth, however, Hutchinson and Paris [51] argue that J is still a controlling parameter. For such situations of J-controlled growth, Paris et al. [52] introduced the concepts of a "tearing modulus" and "J resistance curve" to analyze the stability of such growth. Using the above concepts and the related concepts of CTOA, engineering approaches to elastic-plastic fracture analyses were elaborated upon by Kumar et al. [53] and Kanninen et al. [54].

The mechanics of crack-growth initiation, and substantial amounts of stable growth, in elastic-plastic materials subject to arbitrary load histories,

is not yet understood. This state of affairs is due, in part, to the reason cited by Rice [55] in 1968 that "... no success has been met in attempts to formulate similar general results for incremental plasticity".

Among the first attempts to find a suitable parameter, that is theoretically valid in elastic-plastic fracture mechanics, were those by Bilby [56] and Miyamoto and Kageyama [57] who defined an integral:

$$J_{ext} = \int_{\Gamma} (W^e N_1 - t_i \beta^e_{1i}) ds \tag{II.1}$$

where W^e is the elastic - strain energy density, and β^e_{ij} is the "elastic distortion tensor" such that the increments of elastic displacements are given by: $du^e_i = \beta^e_{\ell i} dx_\ell$. The integral (II.1) is path-independent only for paths on the region of the body that remains elastic, but is path dependent for contours passing through the plastic region. Some studies on J_{ext} were presented by Miyamoto and Kageyama [57,58].

Also, from time to time, ideas of 'energy-balance' and 'energy-release rates', similar to those in the previous section (Section I), are presented in the literature for elastic-plastic materials. However, such ideas of 'energy-release rate' are well known [59,60] to be unworkable for elastic-plastic materials wherein stress saturates to a finite value at large values of strain. In such materials, under quasi-static conditions, it has been shown [59,60,61] that the energy-release rate vanishes [i.e., the value $(\Delta U^*/\Delta a)$ tends to zero when $\Delta a \to o$, where ΔU^* is the total change in global energy due to crack growth by amount Δa]. Of course, the total energy-release for a finite growth step Δa, denoted as $G^{*\Delta}$, remains finite and depends on Δa [60,61]. It is this dependence on the size a that precludes a rational utilization of the "energy-release" concept or the generalization of the original Griffith energy balance concept in elastic-plastic fracture mechanics. Also, the derivation of integrals that may characterize "energy-release" even in finite growth steps along the lines of those in Section I, are no longer possible in elastoplasticity, since the solutions near the crack-tip at time t and at time $(t + \Delta t)$ (during which the crack grows by Δa) are no longer self-similar--due to the elastic unloading that accompanies crack-growth.

On the other hand, the concept of two comparison cracked bodies of identical geometry with crack lengths differing by a differential amount (da), and being identically loaded, is useful in elasto-plastic fracture mechanics. In that process, we abandon the concept of an energy-balance in a single cracked body. Recall that J, in the context of a monotonically loaded cracked body and up to the point of initiation of growth, is simply the total potential energy difference between the cracked-body in question and a comparison cracked-body of identical shape, but with a crack-length differing by (da) from that of the first body, which is loaded monotonically in an identical fashion.

To arrive at fracture criteria that may be theoretically valid in the context of a flow theory of plasticity, studies [5,43] were recently aimed at incremental (or rate) path-independent integral parameters corresponding to an increment (or rate) of externally imposed boundary traction and/or an increment of prescribed displacement. The incremental integral derived in [5], say ΔT_c or $(\dot{T}_c)$ was used in some preliminary studies of creep crack growth [62,63]. However, in (nonlinear) elasticity (and thus, equivalently, in deformation theory of plasticity and monotonic loading), it can be shown that $(\Sigma \Delta T_c)$ (wherein the summation extends over all load increments from beginning to current state) is not equal to J. With slight modifications to the work in [5], attention was later [43] focused on incremental parameters, which under assumptions of nonlinear elasticity (or deformation theory of plasticity) and when integrated over the load path would be equal to J. Two such parameters are elaborately discussed in [43]. One, denoted by (ΔT_p^*) is such that: (i) it is defined as a path independent integral, (ii) it is a direct measure of the crack-tip fields under a flow theory of plasticity, (iii) it is valid for arbitrary loading/unloading histories, and (iv) under conditions of radial monotonic loading when a deformation theory of plasticity may be valid, ($\Sigma\Delta T_p^*$) (with the summation being over the load-path) is equal to J as defined by Eshelby [40] and Rice [50]. The second parameter, (ΔT_p) is such that: (i) it is also defined as a path-independent integral, (ii) it is related to the incremental total potential energy difference between identical and identically loaded cracked bodies with slightly different crack lengths, (iii) it is amenable to measurement on laboratory test specimens by measuring certain strain/displacement data on the external boundary of the specimen, (iv) it is valid for arbitrary loading/unloading histories and is consistent with the use of an incremental flow theory of plasticity, (v) it is such that it is equal to (ΔT_p^*) plus another term involving a volume integral (since this volume integral cannot, in general, be measured directly on a laboratory test specimen, one may use a hybrid numerical-experimental method to evaluate ΔT_p^* from the measured ΔT_p), and finally, (vi) under conditions of radial monotonic loading, ($\Sigma\Delta T_p$) is equal to J as well as ($\Sigma\Delta T_p^*$).

We employ a coordinate system as shown in Fig. 1 and consider all deformations to be infinitesimal. Let at time (or a time-like parameter) τ, the displacement, strain, and stress in the body be u_i, ε_{ij}, and σ_{ij}, respectively. During the time interval τ and $\tau + \Delta\tau$, let the increments in displacement, strain, and stress be Δu_i, $\Delta\varepsilon_{ij}$, and $\Delta\sigma_{ij}$, respectively. The boundary value problem governing the increments, in a dynamic case, can be stated as:

(compatibility): $$\Delta\varepsilon_{ij} = \tfrac{1}{2}(\Delta u_{i,j} + \Delta u_{j,i}) \quad \text{in} \quad V \tag{II.2}$$

(constitutive law): $$\Delta\sigma_{ij} = \partial\Delta V/\partial\Delta\varepsilon_{ij} \tag{II.3}$$

(momentum balance): $$\Delta\sigma_{ij,j} + \Delta f_i = \rho\Delta\ddot{u}_i \;; \quad \Delta\sigma_{ij} = \Delta\sigma_{ji} \tag{II.4}$$

(traction b.c.) $$\Delta\sigma_{ij}n_j = \Delta\bar{t}_i \quad \text{at} \quad S_t \tag{II.5}$$

(displacement b.c.) $$\Delta u_i = \Delta\bar{u}_i \quad \text{at} \quad S_u \tag{II.6}$$

In the context of a classical rate-independent plasticity, the incremental potential ΔV for $\Delta\sigma_{ij}$ may be written as:

$$\Delta V = \tfrac{1}{2}L_{mnpq}\Delta\varepsilon_{mn}\Delta\varepsilon_{pq} - \left(\frac{\alpha}{g}\right)(\lambda_{kl}\Delta\varepsilon_{kl})^2 \tag{II.7}$$

where L_{mnpq} is a tensor of instantaneous elastic modulii, α = 1, or zero according to whether ($\lambda_{kl}\Delta\varepsilon_{kl}$) is positive or negative; λ_{mn} is a tensor normal to the "yield surface" in the stress space, and g is a scalar

related to a measure of strain hardening. We note that α [which determines whether the material particle under question is undergoing elastic loading/unloading (α = o), or whether it is undergoing plastic loading (α = 1)] as well as g are explicit functions of the coordinates x_i; thus, (ΔV) is an explicit function of x_i.

We consider the strain-increments to be decomposed into elastic and plastic parts in the usual fashion, i.e., $\Delta\varepsilon_{ij} = \Delta\varepsilon_{ij}^e + \Delta\varepsilon_{ij}^p$. The incremental stress working density, denoted as ΔW, between times τ and $\tau + \Delta\tau$ is written as:

$$\Delta W = (\sigma_{ij} + \tfrac{1}{2}\Delta\sigma_{ij})\Delta\varepsilon_{ij} \equiv \sigma_{ij}\Delta\varepsilon_{ij} + \Delta V$$

$$= (\sigma_{ij} + \tfrac{1}{2}\Delta\sigma_{ij})(\Delta\varepsilon_{ij}^e + \Delta\varepsilon_{ij}^p) \equiv \Delta W^e + \Delta W^p \tag{II.8}$$

wherein the definitions of ΔW^e and ΔW^p are transparent. Note that ΔW is now an explicit function of x_i, in as much as ΔV and σ_{ij} are explicit functions of x_i.

With the above background, under incremental flow theory of plasticity, the incremental parameter (ΔT_p^*), to serve as an incremental measure of the strength of the crack-tip strain/stress fields, is defined [43] as:

$$\Delta T_p^* = \int_{\Gamma_\varepsilon} [\Delta W N_1 - (t_i + \Delta t_i)\Delta u_{i,1} - \Delta t_i u_{i,1}]ds \tag{II.9}$$

Eq. (II.9) assumes quasi-static conditions. Note also, as in Section I, N_1 is the component of unit normal to Γ_ε along the crack-axis, X_1 (see Fig. 1). Further, (II.9) is considered to be valid (i) under either loading or unloading conditions at the global level as well as near the crack-tip and (ii) for either a stationary crack or for a crack growing stably in a self-similar fashion (along X_1) in which case Γ_ε is a 'small' loop that traverses with the crack-tip.

It is now a simple matter to use the divergence theorem in the region ($V_\Gamma - V_\varepsilon$) and derive an equivalent but "far-field integral" representation for ΔT_p^*, as [43]:

$$\Delta T_p^* = \int_{\Gamma+S_{c\Gamma}} [\Delta W N_1 - (t_i + \Delta t_i)\frac{\partial\Delta u_i}{\partial X_1} - t_i\frac{\partial u_i}{\partial X_1}]ds + \int_{V_\Gamma - V_\varepsilon} [\Delta\sigma_{ij}(\frac{\partial\varepsilon_{ij}}{\partial X_1} + \tfrac{1}{2}\frac{\partial\Delta\varepsilon_{ij}}{\partial X_1}) - \Delta\varepsilon_{ij}(\frac{\partial\sigma_{ij}}{\partial X_1} + \tfrac{1}{2}\frac{\partial\Delta\sigma_{ij}}{\partial X_1})]dV \tag{II.10}$$

In deriving the 'volume integrands' in (II.10), the quasi-static momentum balance conditions (sans Δf_i, as well as $\Delta\ddot{u}_i$) (II.4), and the fact that ΔW explicitly depends on X_1 are used. In specific, it can be shown [43] that:

$$\frac{\partial\Delta V}{\partial X_1}\Big|_{\text{explicit}} = \tfrac{1}{2}\Delta\varepsilon_{ij}\Delta\sigma_{ij,1} - \tfrac{1}{2}\Delta\sigma_{kl}\Delta\varepsilon_{kl,1} \tag{II.11}$$

The integral in (II.10) is path-independent in the sense that when ($\Gamma + S_{c\Gamma}$) and ($V_\Gamma - V_\varepsilon$) in (II.10) are replaced by Γ^* and V* (see Fig. 1), the value of the integral is zero. Note that in the case of growing cracks, Γ in (II.10) is a path fixed in space.

In the case of nonlinear elasticity or a deformation theory of plasticity, the volume integral in (II.10) can be shown [43] to vanish. Thus, the single far-field contour integral of (II.10) also characterizes the crack-tip fields as in (II.9). The motivation for the definition of (ΔT^*_p) as in (II.9) is now clear. Under the conditions: (i) that only a radial monotonic loading exists, for which a deformation theory of plasticity is valid, and (ii) that the crack-tip is stationary, the quantity $(\Sigma\Delta T^*_p)$ [the sum being taken over the load path] is equal to the well-known J. However, the definitions as in (II.9 and II.10) are in general valid for (i) arbitrary loading/unloading histories, as well as (ii) stably growing cracks.

Omitting further details of derivation and motivation, which are elaborated in [43], we now introduce a second incremental path-independent integral, ΔT_p, as:

$$\Delta T_p = \int_S \{-\Delta t_i(\frac{\partial u_i}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta u_i}{\partial X_1}) + n_j(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta \sigma_{ij}}{\partial X_1})\Delta u_i\} dS \qquad \text{(II.12a)}$$

$$= \int_S \{\Delta W N_1 - (t_i + \Delta t_i)\frac{\partial \Delta u_i}{\partial X_1} - \Delta t_i \frac{\partial u_i}{\partial X_1}\} dS \qquad \text{(II.12b)}$$

$$= \int_{\Gamma + S_{c\Gamma}} \{\Delta W N_1 - (t_i + \Delta t_i)\frac{\partial \Delta u_i}{\partial X_1} - \Delta t_i \frac{\partial u_i}{\partial X_1}\} dS + \int_{V-V_\Gamma} [\Delta\varepsilon_{ij}(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\sigma_{ij}}{\partial X_1}) - \Delta\sigma_{ij}(\frac{\partial \varepsilon_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\varepsilon_{ij}}{\partial X_1})] dV \qquad \text{(II.12c)}$$

In the above, S is the external surface of the body, including the crack surface. ΔT_p can be easily shown [43] to be path-independent (i) for arbitrary loading/unloading histories (ii) even when Γ passes through plastic zones, (iii) for either stationary or stably growing cracks, in which case Γ is considered to be a path fixed in space. It can be measured directly using the definition as in (II.12a or b) since these definitions involve only quantities at the external surface of the cracked-body (test specimen). However, in the case of monotonic loading of a stationary crack, ΔT_p as defined in (II.12a) is the incremental (for an increment in loading) difference between the areas under the load-deformation curves for two identical cracked bodies differing in lengths by (da). Thus, it is the incremental total potential energy difference between two identical cracked bodies with slightly different crack lengths, when both bodies are loaded monotonically in an identical fashion. Thus, under conditions for validity of a deformation theory, $\Sigma\Delta T_p \equiv$ J. Note also for arbitrary loading/unloading and with crack growth, the definition of ΔT_p as in (II.12b) makes it formally identical to ΔJ (formally the increment of J) evaluated also on the external boundary of the body. However, ΔJ is no longer path-dependent under arbitrary conditions, while ΔT_p as in (II.12c) is strictly path-independent.

Comparing (II.10) and (II.12c), it is seen:

$$(\Delta T_p) = (\Delta T^*_p) + \int_{V-V_\varepsilon} \{\Delta\varepsilon_{ij}(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\sigma_{ij}}{\partial X_1}) - \Delta\sigma_{ij}(\frac{\partial \varepsilon_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\varepsilon_{ij}}{\partial X_1})\} dv \qquad \text{(II.13)}$$

Of course, once again, under a deformation theory of plasticity (II.13) simplifies to : $\Delta T_p \equiv \Delta T^*_p$. However, in arbitrary flow theory of plasticity and/or with crack growth, the crack-tip parameter ΔT^*_p differs from the (measurable) global parameter (ΔT_p) by a volume integral. Thus, a

hybrid numerical-experimental procedure may be necessary to determine ΔT^*_p as a material property. The complementary representations of ΔT^*_p and ΔT_p can easily be found [43] to be:

$$(\Delta T^*) = \int_{\Gamma_\varepsilon} [(-\Delta W_c N_1 + n_j(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{\partial \Delta\sigma_{ij}}{\partial X_1})\Delta u_i + n_j \frac{\partial \Delta\sigma_{ij}}{\partial X_1} u_i] ds$$

$$= \int_{\Gamma + S_{c\Gamma}} [-\Delta W_c N_1 + n_j(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{\partial \Delta\sigma_{ij}}{\partial X_1})\Delta u_i + n_j \frac{\partial \Delta\sigma_{ij}}{\partial X_1} u_i] ds + \int_{V_\Gamma - V_\varepsilon} [\Delta\sigma_{ij}(\frac{\partial \varepsilon_{ij}}{\partial X_1} + \frac{\partial \Delta\varepsilon_{ij}}{\partial X_1}) - \Delta\varepsilon_{ij}(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{\partial \Delta\sigma_{ij}}{\partial X_1})] dv \qquad \text{(II.14)}$$

where $\Delta W_c = \varepsilon_{ij}\Delta\sigma_{ij} + \frac{1}{2}\Delta\varepsilon_{ij}\Delta\sigma_{ij}$. The representation for ΔT_p is analogous; except that the integral over Γ_ε is replaced by that over the external surface S, and that over $V_\Gamma - V_\varepsilon$ is replaced by the negative of the one over $V - V_\Gamma$, with the integrands remaining the same.

Finally, we present the dynamic counterparts of ΔT^*_p and ΔT_p can be derived [43] to be:

$$\Delta T^*_p = \int_{\Gamma_\varepsilon} [(\Delta W + \Delta T)N_1 - (t_i + \Delta t_i)\frac{\partial \Delta u_i}{\partial X_1} - \Delta t_i \frac{\partial u_i}{\partial X_1}] ds$$

$$= \int_{\Gamma + S_{c\Gamma}} [(\Delta W + \Delta T)N_1 - (t_i + \Delta t_i)\frac{\partial \Delta u_i}{\partial X_1} - \Delta t_i \frac{\partial u_i}{\partial X_1}] ds + \int_{V_\Gamma - V_\varepsilon} [\Delta\sigma_{ij}(\frac{\partial \varepsilon_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\varepsilon_{ij}}{\partial X_1}) - \Delta\varepsilon_{ij}(\frac{\partial \sigma_{ij}}{\partial X_1} + \frac{1}{2}\frac{\partial \Delta\sigma_{ij}}{\partial X_1}) + \rho(\ddot{u}_i + \Delta\ddot{u}_i)\frac{\partial \Delta u_i}{\partial X_1} - \rho(\dot{u}_i + \Delta\dot{u}_i)\frac{\partial \Delta\dot{u}_i}{\partial X_1} + \rho\Delta\ddot{u}_i \frac{\partial u_i}{\partial X_1} - \rho\Delta\dot{u}_i \frac{\partial \dot{u}_i}{\partial X_1}] dv \qquad \text{(II.15)}$$

It can easily be shown [43] that, in the case of elastic materials, the summation of ΔT^*_p of (II.15) over the time history, is equal to J' of (I.16). Also $\Delta T = \rho(\dot{u}_i + \frac{1}{2}\Delta\dot{u}_i)\Delta\dot{u}_i$. The representation for (ΔT_p) in the dynamic case is analogous to that in (II.15), except that, while keeping the respective integrands the same, the integral over Γ_ε is replaced by that over the external boundary S, and the one over $V_\Gamma - V_\varepsilon$ is replaced by the negative of that over $V - V_\Gamma$.

We present now some applications of the J', ΔT^*_p, and ΔT_p integrals in elastic and elasto-plastic dynamic fracture mechanics.

APPLICATIONS TO ELASTIC-PLASTIC DYNAMIC FRACTURE

We consider the problem of a center-cracked specimen subject to a uniaxial tensile pulse of the Heavyside step-function type, as illustrated in Fig. 3, wherein the properties of the material, modeled herein as being elastic-perfectly-plastic, are also indicated. The problem is analyzed for three values of the ratio of applied stress (σ_o) to the yield stress (σ_{ys}): (σ_o/σ_{ys}) = 0.0; 0.25; 0.5. Thus, the first case corresponds to linear elasticity. The symmetry of geometry and loading allow one to analyze only a quarter of the specimen, which is modelled by finite elements as shown in Fig. 4. Results are presented here only for the case of a stationary crack for space reasons; while those for a propagating crack are to be included in the presentation at ICF6. The computed shapes of plastic zones at various instants of time are shown in Figs. 5 and 6 for the cases (σ_o/σ_{ys}) = 0.25 and 0.5, respectively. The crack-surface deformation profiles for the cases (σ_o/σ_{ys}) = 0.0 and 0.5 are shown in Fig. 7 and 8, respectively. The values of $\Sigma\Delta T_p^* = \Sigma\Delta T$ (crack-tip), at various instants of time, and for various paths are shown, normalized w.r.t. the static J value, for the case (σ_o/σ_{ys}) = 0 in Fig. 9, while similar results are shown for the case (σ_o/σ_{ys}) = 0.5 in Fig. 10. These results indicate the relative path-independence of the computed ΔT_p^* values both in elastic as well as elastic-plastic cases. The variation of computed k_I <which is evaluated from $\Sigma\Delta T_p^*$ for the elastic case, using the relation between k_I and $J' = \Sigma\Delta T_p^*$ for the elastic case given in [8]> with time for the case (σ_o/σ_{ys}) = 0 is shown in Fig. 11; this result agrees with the analytical result of Baker [64]. The nomenclature (D_m), (D_mD_c), etc. indicate the time of arrival of various waves at the crack-tip, as explained in [65]. The variation of $\Sigma\Delta T_p^*$ for the elastic case, (σ_o/σ_{ys}) = 0, can be seen to be linear in time from Fig. 12. The variations w.r.t time of $\Sigma\Delta T_p^*$, for the elastic-plastic cases, (σ_o/σ_{ys}) = 0.25 and 0.5, respectively, are shown in Figs. 13 and 14, from which it can be seen that the strength of the crack-tip field, i.e. $\Sigma\Delta T_p^*$, is lower than that in the elastic case. On the other hand, the variations of the global parameter $(\Sigma\Delta T_p)$ for the cases (σ_o/σ_{ys}) = 0 and 0.5 are shown in Fig. 15. The variations of the crack-opening stretch with time for the three cases, (σ_o/σ_{ys}) = 0, 0.25, and 0.5, are shown in Fig. 16. From Fig. 17 it can be seen that $J' = (\Sigma\Delta T_p^*)$, in linear elasto-dynamics, varies quadratically w.r.t the crack-opening stretch; while Fig. 18 shows that in the elastic-plastic case, (σ_o/σ_{ys}) = 0.5, the crack-tip parameter $\Sigma\Delta T_p^*$ varies linearly w.r.t δ. This is interesting and analogous to the well-known result in quasi-static case, viz., the J integral is linearly proportional to the crack-opening stretch [66]. The above results serve to illustrate the validity of ΔT_p^* as a crack-tip parameter. Its use as a fracture parameter requires further validation, and additional results concerning this are currently being generated.

ACKNOWLEDGEMENTS

This work was supported by the National Science Foundation under Grant MEA-8306359, ONR under Contract N00014-78-C-0636, and by NASA under Contract NAG 346. These supports are gratefully acknowledged. The assistance of Ms. J. Webb in the preparation of this manuscript is sincerely appreciated.

REFERENCES

[1] Rice, J. R., "The Mechanics of Fracture", (Ed. F. Erdogan), ASME-AMD Vol. 19, (1976), 23.
[2] Bilby, B. A., "Fracture 1977", (Ed. D. M. R. Taplin) University of Waterloo Press, Canada, IV , (1977), 1.
[3] Truesdell, C. and Noll, W., "The Nonlinear Field Theories of Mechanics", Handbuch der Physik III/3, Springer, Berlin, (1965).
[4] Atluri, S. N., Computers and Structures, 18 , (1984), 93.
[5] Atluri, S. N., Engineering Fracture Mechanics, 16 , (1982), 341.
[6] Atkinson, C. and Eshelby, J. D., Int. J. Fract. Mech., 4 , (1968), 3.
[7] Eshelby, J. D., "Inelastic Behavior of Solids", (Ed. M. F. Kanninen, et al.), McGraw-Hill, New York, (1970), 77.
[8] Nishioka, T. and Atluri, S. N., Engineering Fracture Mechanics, 18 , (1983), 1.
[9] Nishioka, T. and Atluri, S. N., "Dynamic Crack Propagation Analysis Using a New Path-Independent Integral and Moving Isoparametric Elements", AIAA Journal, (In Press), (1983).
[10] Nishioka, T. and Atluri, S. N., "On the Computation of Mixed-Mode K-Factors for a Dynamically Propagating Crack, Using Path-Independent Integrals J'", Engineering Fracture Mechanics, (In Press).
[11] Budiansky, B. and Rice, J. R., J. Appl. Mech., 40 , (1973), 201.
[12] Herrmann, A-G. and Herrmann, G., J. Appl. Mech., 48 , (1981), 525.
[13] Atluri, S. N. and Nishioka, T., "Path-Independent Integrals in Dynamic Fracture", Proc. Workshop on Dynamic Fracture, Caltech, Pasadena, 16-17 February 1983, (In Press).
[14] Kishimoto, K., Aoki, S., and Sakata, M., Engineering Fracture Mechanics, 13 , (1980), 841.
[15] Strifors, H., Int. J. Solids Structures, 10 , (1974), 1389.
[16] Carlsson, A. J., "Prospects of Fracture Mechanics", (Ed. G. C. Sih et al.), Noordhoff Int., Leyden, (1974), 139.
[17] Freund, L. B., J. of Elasticity, 4 , (1972), 341.
[18] Sih, G. C., Inelastic Behavior of Solids, (Eds. M. F. Kanninen, et al.), McGraw-Hill, (1970), 607.
[19] Atluri, S. N. and Nishioka, T., "Energy-Release Rates in Dynamic Fracture: Path Invariant Integrals and Some Computational Studies", Proc. Int. Conf. on Fract. Mech. Applied to Mat. Evaluation and Structure Design, Melbourne, Australia, (1982).
[20] Nishioka, T. and Atluri, S. N., Engineering Fracture Mechanics, 17 , (1983), 23.
[21] Bui, H. D., "Fracture 1977", (Ed. D. M. R. Taplin), University of Waterloo Press, Canada, III , (1977), 91.
[22] Bui, H. D. and Erlacher, A., In: "Fracture 1981, ICF5", (Ed. D. Francois), 2 , (1981), 533.
[23] Erlacher, A., "Fracture 1981, ICF5", (Ed. D. Francois), 5, (1981), 2187.
[24] Irwin, G. R., J. Appl. Mech., 24 , (1957), 361.
[25] Erdogan, F., "Fracture II", (Ed. H. Liebowitz), Academic Press, (1968), 498.
[26] Gurtin, M. E. and Yatomi, C., Arch. for Rat. Mech. and Analy., 74 , (1980), 231.
[27] Achenbach, J. D., Mechanics Today, Vol. 1, (Ed. S. Nemat-Nasser), Pergamon, (1972), 1.
[28] Yatomi, C., Int. J. Fracture, 18 , (1982), 233.
[29] Kishimoto, K., Aoki, S., and Sakata, M., Engg. Fract. Mech., 13 , (1981), 387.
[30] Ouyang, C., Int. J. Nonlinear Mech., 18 , (1983), 79.
[31] Aoki, S., Kishimoto, K., and Sakata, M., J. Appl. Mech., 48 ,(1981), 825.
[32] Nilsson, F., Int. J. Solids Structures, 9 , (1973), 1107.
[33] Gurtin, M. E., Int. J. Fracture, 12 , (1976), R643.
[34] Noether, E., Gottinger Nachrichten (Math.-Phys. Klasse), (1918), 235, (English Transl. by M. A. Tavel), Transport Theory and Statistical Physics, 1 , (1971), 183.
[35] Gunther, W., Abh. braunsch. wisch. Ges., 14 , (1962), 54.
[36] Knowles, J. K. and Sternberg, E., Arch. Rat. Mech. Analy., 44 , (1972), 187.
[37] Fletcher, D. C., Arch. Rat. Mech. Analy., 60 , (1976), 329.
[38] Golebiewska-Herrmann, A., Int. J. Solids Structures, 17 , (1981), 1.
[39] Eshelby, J. D., Phil. Trans., A244 , (1951), 81.
[40] Eshelby, J. D., Solid State Physics, 3 , Academic Press, (1951), 79.
[41] Golebiewska-Herrmann, A., Int. J. Solids Struct., 18 , (1982), 319.
[42] Bui, H. D., Engineering Fracture Mech., 6 , (1974), 257.

[43] Atluri, S. N., Nishioka, T., and Nakagaki, M., "Incremental Path-Independent Integrals in Inelastic and Dynamic Fracture Mechanics", Rept. GIT-CACM-SNA-83-27, Georgia Tech, May 1983, 82 pages. Also Engineering Fracture Mechanics, (In Press).

[44] Hutchinson, J. W., J. Mech. Phys. Solids, 35 , (1968), 13.

[45] Rice, J.R. and Rosengren, G.F., J. Mech. Phys. Solids, 16, (1968), 1.

[46] Begley, J. A. and Landes, J. D., "Fracture Toughness", ASTM STP 514, (1972), 1.

[47] Rice, J. R., Paris, P. C., and Merkle, J. G., "Progress in Flaw Growth and Fracture Toughness Testing", ASTM STP 536, (1973), 231.

[48] Bucci, R. J., Paris, P. C., Landes, J. D., and Rice, J. R., "Fracture Toughness", ASTM STP 514, (1972), 40.

[49] Shih, C. F. and Hutchinson, J. W., J. Eng. Mat. and Tech., 98 , (1976), 289.

[50] Rice, J. R., J. Appl. Mech., 35 , (1968), 376.

[51] Hutchinson, J. W. and Paris, P. C., "Elastic-Plastic Fracture", ASTM STP 668, (1979), 57.

[52] Paris, P. C. and Zahalak, G. I., "Nonlinear and Dynamic Fracture Mechanics", (Eds. N. Perrone and S. N. Atluri), ASME, (1979), 125.

[53] Kumar, V. German, M. D., and Shih, C. F., EPRI-NP-1931, Electric Power Research Institute, Palo Alto, California, (1981).

[54] Kanninen, M. F., et al., EPRI NP-1734, Electric Power Research Institute, Palo Alto, California, (1981).

[55] Rice, J. R., "Fracture, Vol. II", (Ed. H. Liebowitz), Acadmic Press, (1968), 213.

[56] Bilby, B. A., ICF3, Munich, Part XI, (1973), 1.

[57] Miyamoto, H. and Kageyama, K., "Proc. Int. Conf. Num. Meth. Fract. Mech.", Swansea, (1978), 479.

[58] Miyamoto, H. and Kageyama, K., "Recent Research on Mechanical Behavior of Solids", University of Tokyo Press, (1979), 229.

[59] Rice, J. R., "Proc. Ist. Int. Congress on Fracture", Sendai, (Ed. T. Yokobori, et al.), 1 , (1966), 309.

[60] Nakagaki, M., Chen, W. H., and Atluri, S. N., "Elastic-Plastic Fracture", ASTM STP 668, (1979), 195.

[61] Kfouri, A. P. and Rice, J. R., "Fracture 1977", (Ed. D. M. R. Taplin), University of Waterloo Press, (1977), 43.

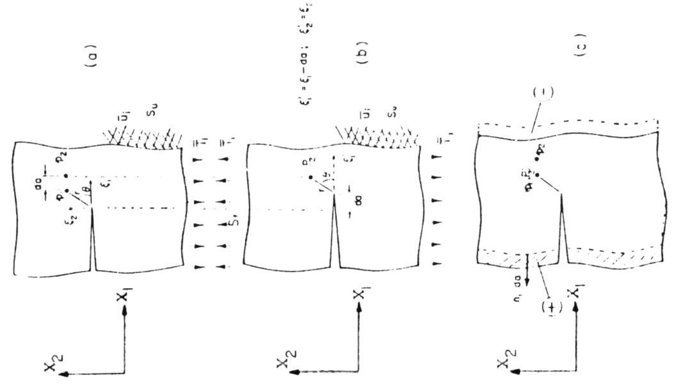
[62] Stonesifer, R. B. and Atluri, S. N., Engineering Fracture Mechanics, 16 , (1982), 625.

[63] Stonesifer, R. B. and Atluri, S. N., Engineering Fracture Mechanics, 16 , (1982), 769.

[64] Baker, B. R., J. Appl. Mech., 29 , (1962), 449.

[65] Nishioka, T. and Atluri, S. N., J. Appl. Mech., 47 , (1980), 577.

[66] Atluri, S. N., Nakagaki, M., and Chen, W. H., Press. Vessel Tech. Part II (Proc. III ICPVT, Tokyo), ASME, New York, (1977), 579.



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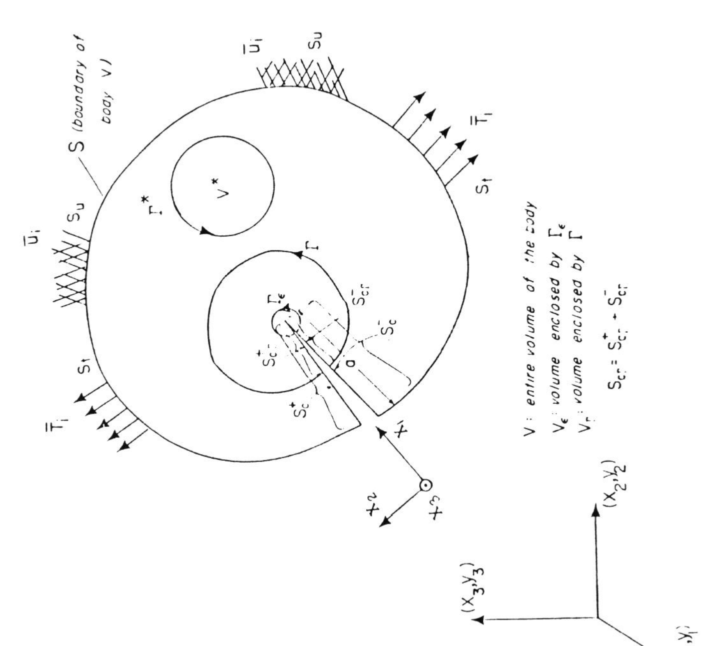
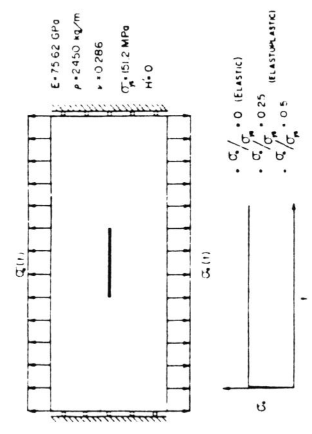


Figure 1



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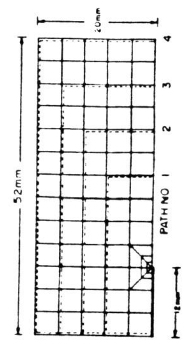

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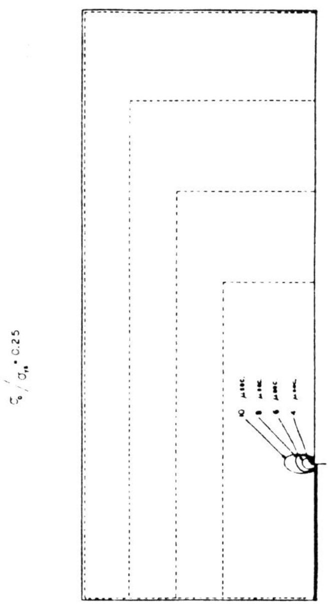

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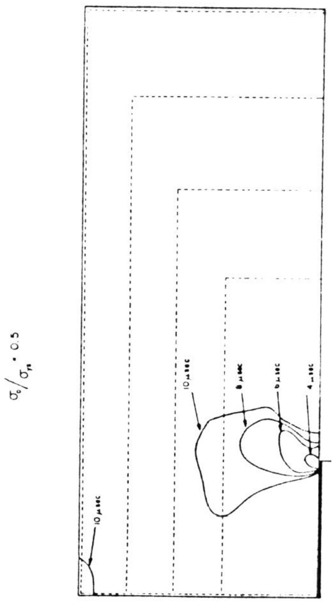

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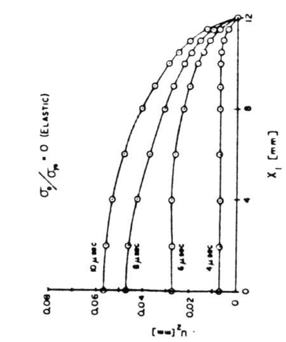

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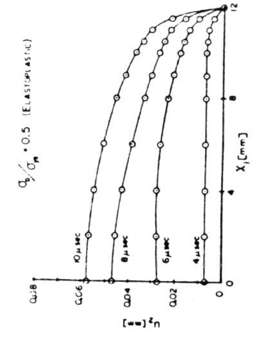

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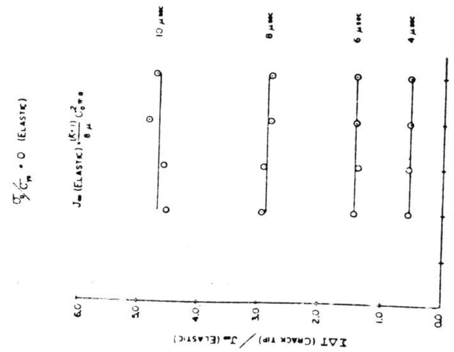

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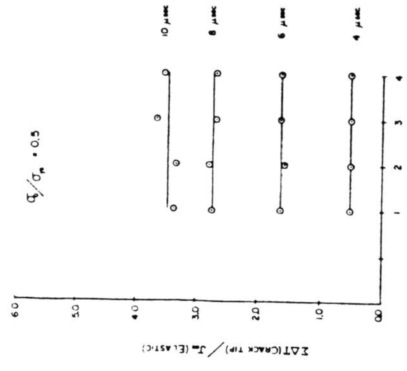

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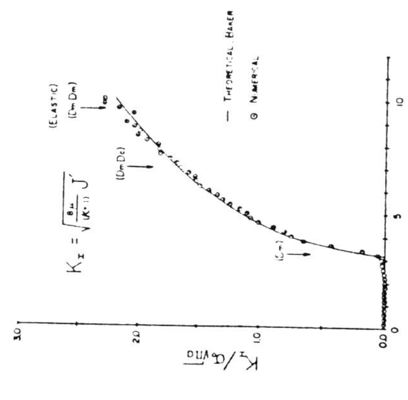

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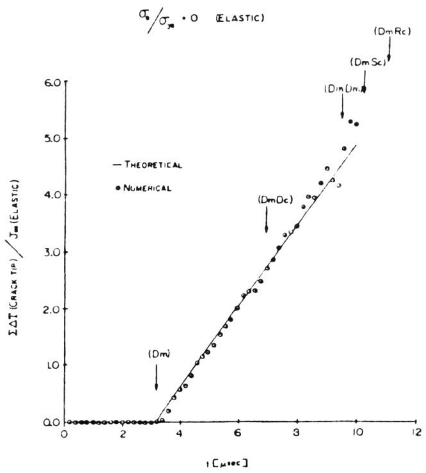

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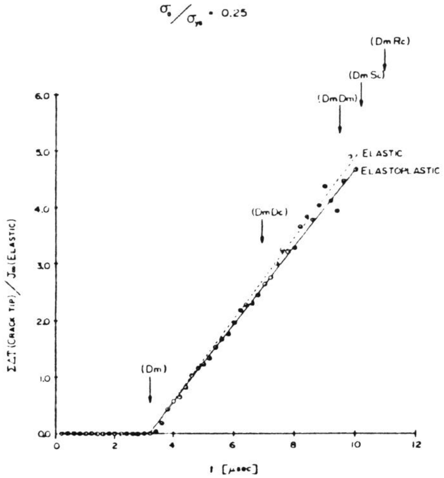

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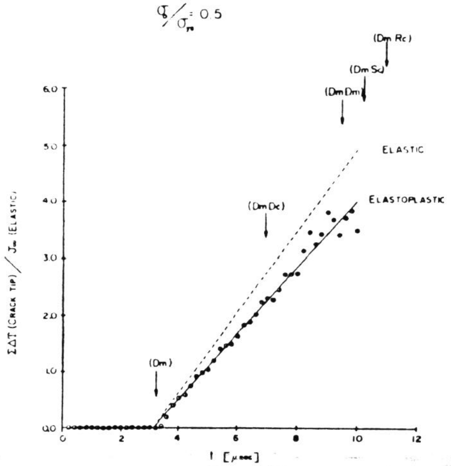

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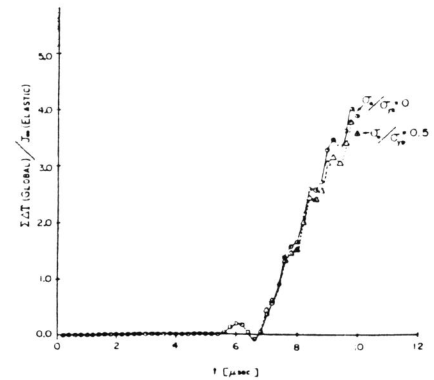

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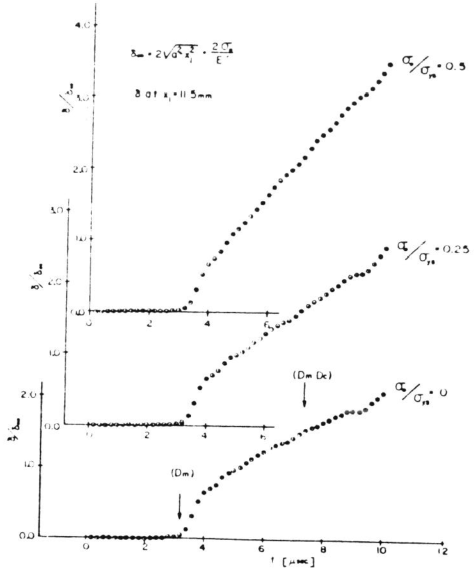

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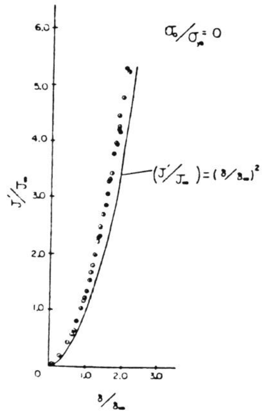

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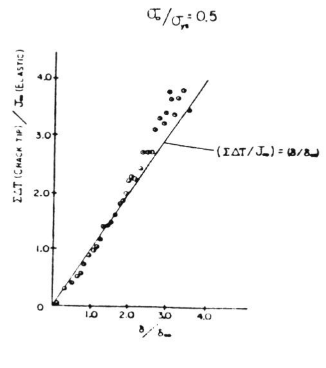

Figure 18