STRESS INTENSITY FACTORS IN PIPE ELBOWS

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ABSTRACT

The general problem of toroidal shells containing a symmetrically located through crack is formulated by using Reissner's shell theory. For symmetric loading conditions the problem is reduced to a pair of singular integral equations. The solution is obtained for positive and negative curvature ratios and the stress intensity factors are calculated.

KEYWORDS

Cracks in shells; pipe elbows; stress intensity factors; fracture of pipes; pressure vessels; toroidal shells.

INTRODUCTION

In elastic shells representing such structural components as relatively thin-walled pressure vessels, storage tanks, and pipes the curvature is known to have a considerable magnifying effect on the stress intensity factors at the tips of a through crack which may exist in the shell wall. As in flat plates, mostly because of mathematical expediency the early studies of crack problems in shells were based on the classical shallow shell theory and were confined to cylinders and spheres containing a meridional crack (see, for example, Erdogan, 1977 for review and references). However, these classical solutions were known to contain certain inconsistencies with regard to the asymptotic stress field around the crack tips which, in flat plates were removed by using Reissner's theory (Knowles and Wang, 1960). In recent years similar discrepancies between the asymptotic results given by the classical shell theory and those obtained from the plane theory of elasticity have also been removed by using a higher order shell theory which is consistent with the number of independent physical boundary conditions (see Krenk, 1978; Sih and Hagendorf, 1977; Delale and Erdogan, 1979a, Delale and Erdogan 1979b, for cylindrical and spherical shells). Again, in all these studies the shell was assumed to be either cylindrical or spherical. Needless to say, some of the important applications of the crack problems in shells may be in components such as pipe elbows and other toroidal shells in which there are more than one distinct nonzero curvature. In this paper the general problem of a shell with two nonzero curvatures having a through crack in one of the principal planes of curvatures is considered by using Reissner's theory (Reissner and Wan, 1969). Specific results are obtained for a pipe elbow or torus with either positive or negative curvature ratio. The asymptotic results are shown to be consistent with the in-plane elasticity solutions. Using this fact, an expression for the strain energy release rate in elastic shells is derived.

BASIC FORMULATION

Consider the shallow shell shown in Fig. 1. Let $Z = Z(x_1, x_2)$ be the equation of the middle surface. Assume that the curvatures

$$(1/R_1) = -\partial^2 Z/\partial x_1^2 \;,\; (1/R_2) = -\partial^2 Z/\partial x_2^2 \;,\; (1/R_{12}) = -\partial^2 Z/\partial x_1 \partial x_2 \tag{1}$$

are all nonzero and distinct. Referring to Fig. 1 for notation, to Appendix A for the normalized quantities, and to (Delale and Erdogan, 1979a) for the derivation, the governing equations of the shell may be expressed as follows:

$$\nabla^4 \phi - \frac{1}{\lambda^2}\left(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2}\right) w = 0 \;, \tag{2}$$

$$\nabla^4 w + \lambda^2 (1-\kappa\nabla^2)\left(\lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2}\right) \phi = 0 \;, \tag{3}$$

$$\kappa \nabla^2 \psi - \psi - w = 0 \;,\; \kappa \frac{1-\nu}{2} \nabla^2 \Omega - \Omega = 0 \tag{4,5}$$

where

$$\psi(x,y) = \kappa\left(\frac{\partial \beta_x}{\partial x} + \frac{\partial \beta_y}{\partial y}\right) - w \;,\; \Omega(x,y) = \frac{\partial \beta_x}{\partial y} - \frac{\partial \beta_y}{\partial x} \;, \tag{6,7}$$

ϕ is the stress function, w is the z-component of the displacement vector, β_x and β_y are the components of the rotation (of the normal to the middle surface), and the remaining quantities are defined in Appendix A. In this paper we consider only the perturbation problem in which the crack surface tractions are the only external loads acting on the shell. Defining

$$\nabla_\lambda^2 = \lambda_1^2 \frac{\partial^2}{\partial y^2} - 2\lambda_{12}^2 \frac{\partial^2}{\partial x \partial y} + \lambda_2^2 \frac{\partial^2}{\partial x^2} \tag{8}$$

and eliminating w, (2) and (3) may be reduced to

$$\nabla^4 \nabla^4 \phi + (1 - \kappa\nabla^2)\, \nabla_\lambda^2 \nabla_\lambda^2 \phi = 0 \,. \tag{9}$$

Let the solution of (9) be of the form

$$\phi(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} g(x,\alpha) e^{-i\alpha y}\, d\alpha \,. \tag{10}$$

Substituting (10) into (9) we obtain an 8th order differential equation for g. Looking for the solution of this differential equation which is of the form e^{mx} we obtain the following characteristic equation:

$$m^8 - (4\alpha^2 + \kappa\lambda_2^4)m^6 - 4i\kappa\alpha\lambda_2^2\lambda_{12}^2 m^5 + (6\alpha^4 + \lambda_2^4 + 4\kappa\lambda_{12}^4\alpha^2 + 2\alpha^2\kappa\lambda_1^2\lambda_2^2$$
$$+ \alpha^2\lambda_2^4\kappa)m^4 + 4i\lambda_{12}^2(\alpha\lambda_2^2 + \kappa\alpha^3\lambda_1^2 + \kappa\alpha^3\lambda_2^2)m^3 - (4\alpha^4 + 4\lambda_{12}^4 + 2\lambda_1^2\lambda_2^2$$
$$+ \kappa\lambda_1^4\alpha^2 + 4\kappa\lambda_{12}^4\alpha^2 + 2\kappa\lambda_1^2\lambda_2^2\alpha^2)\alpha^2 m^2 - (1 + \kappa\alpha^2)4i\alpha^3\lambda_1^2\lambda_{12}^2 m$$
$$+ \alpha^4(\alpha^4 + \lambda_1^4 + \kappa\lambda_1^4\alpha^2) = 0. \tag{11}$$

If we now assume that the axes x_1 and x_2 lie in the principal planes of curvature, $\lambda_{12} = 0$ and (11) becomes a fourth degree equation in m^2. Furthermore, defining

$$p = m^2 - \alpha^2 \tag{12}$$

equation (11) becomes

$$p^4 - \kappa\lambda_2^4 p^3 + (2\kappa\lambda_1^2\lambda_2^2\alpha^2 - 2\kappa\lambda_2^4\alpha^2 + \lambda_2^4)p^2$$
$$+ (2\kappa\lambda_1^2\lambda_2^2\alpha^4 - \kappa\lambda_2^4\alpha^4 - \kappa\lambda_1^4\alpha^4 + 2\lambda_2^4\alpha^2 - 2\lambda_1^2\lambda_2^2\alpha^2)p$$
$$+ (\lambda_1^2 - \lambda_2^2)^2\alpha^4 = 0. \tag{13}$$

Let the roots of (11) obtained from (12) and (13) be $m_1, \ldots, m_8$ which are ordered as

$$\mathrm{Re}(m_j) < 0 \;,\; m_{j+4} = -m_j \;,\; j = 1,\ldots,4 \,. \tag{14}$$

Then the unknown function $g(x,\alpha)$ given in (10) may be expressed as

$$g(x,\alpha) = \begin{cases} \sum_1^4 R_j(\alpha)\, e^{m_j x} \;, & x > 0 \\ \sum_5^8 R_j(\alpha) e^{m_j x} \;, & x < 0 \,. \end{cases} \tag{15}$$

Also, assuming $w(x,y)$ of the form

$$w(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x,\alpha) e^{-i\alpha y}\, d\alpha \;, \tag{16}$$

from (2), (10) and (15) we find

$$f(x,\alpha) = \begin{cases} \lambda^2 \sum_1^4 \frac{p_j^2 R_j(\alpha)}{\lambda_2^2 m_j^2 - \lambda_1^2\alpha^2}\, e^{m_j x} \;, & x > 0 \\ \lambda^2 \sum_5^8 \frac{p_{j-4}^2 R_j(\alpha)}{\lambda_2^2 m_j^2 - \lambda_1^2\alpha^2}\, e^{m_j x} \;, & x < 0 \end{cases} \tag{17}$$

Similarly, assuming the functions Ω and ψ of the form

$$\Omega(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} h(x,\alpha)e^{-i\alpha y}d\alpha\ ,\quad \psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\theta(x,\alpha)e^{-i\alpha y}d\alpha\ , \tag{18,19}$$

from (4), (5), (16) and (17) we obtain

$$h(x,\alpha) = \begin{cases} A_1(\alpha)e^{r_1 x}\ , & x > 0 \\ A_2(\alpha)e^{r_2 x}\ , & x < 0 \end{cases} \tag{20}$$

$$\theta(x,\alpha) = \begin{cases} \lambda^2 \sum_1^4 \dfrac{p_j^2 R_j(\alpha)}{(\lambda_2^2 m_j^2 - \lambda_1^2\alpha^2)(\kappa p_j - 1)} e^{m_j x}\ , & x > 0 \\ \lambda^2 \sum_5^8 \dfrac{p_{j-4}^2 R_j(\alpha)}{(\lambda_2^2 m_j^2 - \lambda_1^2\alpha^2)(\kappa p_{j-4}^2 - 1)} e^{m_j x}\ , & x < 0 \end{cases} \tag{21}$$

where

$$r_1 = -\left[\alpha^2 + \frac{2}{\kappa(1-\nu)}\right]^{\frac{1}{2}}\ ,\quad r_2 = \left[\alpha^2 + \frac{2}{\kappa(1-\nu)}\right]^{\frac{1}{2}}\ . \tag{22}$$

After determining the functions ψ and Ω the components of the rotation vector may be obtained as

$$\beta_x = \frac{\partial\psi}{\partial x} + \kappa\frac{1-\nu}{2}\frac{\partial\Omega}{\partial y}\ ,\quad \beta_y = \frac{\partial\psi}{\partial y} - \kappa\frac{1-\nu}{2}\frac{\partial\Omega}{\partial x}\ . \tag{23}$$

Let us now assume that $x_1 = 0$ is a plane of symmetry with respect to the external loads as well as the shell geometry. Thus, the stress and moment resultants which appear in boundary conditions satisfy the following symmetry conditions:

$$N_{xx}(x,y) = N_{xx}(-x,y)\ ,\ N_{xy}(x,y) = -N_{xy}(-x,y)\ ,\ V_x(x,y) = -V_x(-x,y),$$

$$M_{xx}(x,y) = M_{xx}(-x,y)\ ,\ M_{xy}(x,y) = -M_{xy}(-x,y)\ , \tag{24}$$

and it is sufficient to consider one half (i.e., x>0) of the shell only. Note that the stress and moment resultants necessary to express the boundary conditions on the crack surfaces are given by (Fig. 1).

$$N_{xx} = \frac{\partial^2\phi}{\partial y^2}\ ,\ N_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}\ ,\ \frac{\partial V_x}{\partial y} = \frac{\partial^2 w}{\partial x\partial y} + \frac{\partial\beta_x}{\partial y}\ ,$$

$$M_{xx} = \frac{a}{h\lambda^4}\left(\frac{\partial\beta_x}{\partial x} + \nu\frac{\partial\beta_y}{\partial y}\right)\ ,\ M_{xy} = \frac{a}{h\lambda^4}\frac{1-\nu}{2}\left(\frac{\partial\beta_x}{\partial y} + \frac{\partial\beta_y}{\partial x}\right)\ , \tag{25}$$

By using the solution given above for x>0 we find

$$N_{xx}(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\alpha^2\sum_1^4 R_j(\alpha)e^{m_j x - iy\alpha}d\alpha\ , \tag{26}$$

$$N_{xy}(x,y) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\alpha\sum_1^4 m_j R_j(\alpha)e^{m_j x - iy\alpha}d\alpha\ , \tag{27}$$

$$M_{xx}(x,y) = \frac{a}{h\lambda^4}\frac{1}{2\pi}\int_{-\infty}^{\infty}\lambda^2\sum_1^4\frac{p_j^2(m_j^2-\nu\alpha^2)R_j(\alpha)}{(\lambda_2^2 m_j^2-\lambda_1^2\alpha^2)(\kappa p_j-1)}e^{m_j x - iy\alpha}d\alpha$$

$$-\frac{i}{2\pi}\frac{a\kappa}{2h\lambda^4}(1-\nu)^2\int_{-\infty}^{\infty}\alpha\, r_1 A_1(\alpha)e^{r_1 x - iy\alpha}d\alpha\ , \tag{28}$$

$$M_{xy}(x,y) = -\frac{i}{2\pi}\frac{a(1-\nu)}{h\lambda^4}\int_{-\infty}^{\infty}\lambda^2\alpha\sum_1^4\frac{p_j^2 m_j R_j(\alpha)}{(\lambda_2^2 m_j^2-\lambda_1^2\alpha^2)(\kappa p_j-1)}e^{m_j x - iy\alpha}d\alpha\ ,$$

$$-\frac{1}{2\pi}\frac{a\kappa(1-\nu)^2}{4h\lambda^4}\int_{-\infty}^{\infty}A_1(\alpha)(r_1^2+\alpha^2)e^{r_1 x - iy\alpha}d\alpha\ , \tag{29}$$

$$\frac{\partial V_x(x,y)}{\partial y} = -\frac{i}{2\pi}\int_{-\infty}^{\infty}\lambda^2\alpha\kappa\sum_1^4\frac{p_j^3 m_j R_j(\alpha)}{(\lambda_2^2 m_j^2-\lambda_1^2\alpha^2)(\kappa p_j-1)}e^{m_j x - iy\alpha}d\alpha$$

$$-\frac{\kappa(1-\nu)}{4\pi}\int_{-\infty}^{\infty}\alpha^2 A_1(\alpha)e^{r_1 x - iy\alpha}d\alpha\ . \tag{30}$$

It is seen that the problem is solved once the unknown functions $R_1,\dots,R_4$, and A_1 are determined.

THE INTEGRAL EQUATIONS

The formulation of the perturbation problem given above for x>0 is equivalent to a tenth order system and satisfies the regularity conditions at $x^2+y^2 = \infty$. Thus, the formulation contains only five unknown functions $R_1,\dots,R_4$, and A_1 which are determined from the following boundary conditions (see (24) and Fig. 1):

$$N_{xy}(0,y) = 0,\ M_{xy}(0,y) = 0,\ V_x(0,y) = 0,\ -\infty<y<\infty\ , \tag{31a-c}$$

$$\lim_{x\to+0} N_{xx}(x,y) = F_1(y)\ ,\ |y| < 1,\ u(+0,y) = 0,\ 1<|y|<\infty, \tag{32a,b}$$

$$\lim_{x\to+0} M_{xx}(x,y) = F_2(y),\ |y| < 1,\ \beta_x(+0,y) = 0,\ 1<|y|<\infty \tag{33a,b}$$

where F_1 and F_2 are known functions. The three homogeneous conditions (31) may be used to eliminate three of the five unknown functions; the mixed boundary conditions (32) and (33) may then give a system of dual integral equations to determine the remaining two functions. The problem may also be reduced to a system of singular integral equations in terms of the following new unknown functions which are the natural complements of the specified crack surface loads F_1 and F_2:

$$G_1(y) = \frac{\partial}{\partial y} u(+0,y) \ , \ G_2(y) = \frac{\partial}{\partial y} \beta_x(+0,y) . \qquad (34a,b)$$

Thus, expressing $R_1,\dots,R_4$, and A_1 in terms of G_1 and G_2 and substituting into (32a) and (33a), after some lengthy analysis the integral equations may be expressed as

$$\sum_{1}^{2} \int_{-1}^{1} \left[\frac{\delta_{ij} a_j}{t-y} + k_{ij}(t,y) \right] G_j(t)dt = 2\pi\, F_i(y), \ |y|<1, \ i=1,2 \ , \qquad (35)$$

where $a_1=1$, $a_2=a(1-\nu^2)/h\lambda^4$ and k_{ij} are known functions. From (32b), (33b) and (34) it follows that (35) must be solved under the following single-valuedness conditions:

$$\int_{-1}^{1} G_j(t)dt = 0, \ j = 1,2 \ . \qquad (36)$$

THE SOLUTION AND RESULTS

The solution of (35) is of the following form which may be obtained numerically in a straightforward manner:

$$G_i(y) = g_i(y)/(1-y^2)^{\frac{1}{2}} , \ i = 1,2 \qquad (37)$$

After performing an asymptotic analysis the stress intensity factors may be expressed in terms of $g_i(\mp 1)$, $(i=1,2)$. For example, referring to (Delale and Erdogan, 1979a) for details, for a uniform membrane loading on the crack surface $N_{11}(0,x_2) = -h\sigma_m$, $M_{11}(0,x_2) = 0$, we obtain

$$k_{mm} = \frac{k_1(0)}{\sigma_m\sqrt{a}} = -\frac{E}{2\sigma_m} g_1(1), \ k_{bm} = \frac{k_1(h/2)-k_1(0)}{\sigma_m\sqrt{a}} = -\frac{Eh}{4a\sigma_m} g_2(1) \qquad (38)$$

where k_{mm} and k_{bm} are the membrane and bending components of the Mode I stress intensity factor ratio $k_1(x_3)/\sigma_m\sqrt{a}$. Similarly, for the external loads $N_{11}(0,x_2) = 0$, $M_{11}(0,x_2) = -h^2\sigma_b/6$, $-a<x_2<a$, membrane and bending components of the stress intensity factor ratio $k_1(x_3)/\sigma_b\sqrt{a}$ may be expressed as

$$k_{mb} = \frac{k_1(0)}{\sigma_b\sqrt{a}} = -\frac{E}{2\sigma_b} g_1(1), \ k_{bb} = \frac{k_1(h/2)-k_1(0)}{\sigma_b\sqrt{a}} = -\frac{Eh}{4a\sigma_b} g_2(1) \ . \qquad (39)$$

Note that the Mode I stress intensity factor is given by

$$k_1(x_3) = k_m + k_b(x_3), \ k_m = k_{mj}\sigma_j\sqrt{a}, \ k_b(x_3) = k_{bj}\sigma_j\sqrt{a}\,\frac{x_3}{h/2}, \ j = m,b, \qquad (40)$$

where k_m and k_b are the membrane and bending stress intensity factors. Using these results the strain energy release rate G (per unit crack extension at one crack tip) may be evaluated as

$$G = \frac{\pi h}{E}(k_m^2 + k_b^2/3). \qquad (41)$$

Table 1. Stress Intensity Factor Ratios, a=10h, ν=0.3

R_1/R_2 \ $a/\sqrt{hR_2}$	0.1	0.25	0.50	0.75	1.0	1.5	2.0
				k_{mm}			
0.2	1.040	1.216	1.681	2.246	2.854	4.145	5.556
1/3	1.025	1.141	1.471	1.897	2.378	3.434	4.590
0.5	1.018	1.100	1.349	1.686	2.078	2.969	3.965
2	1.005	1.035	1.130	1.277	1.460	1.917	2.471
3	1.004	1.027	1.101	1.219	1.368	1.740	2.195
5	1.003	1.021	1.078	1.168	1.286	1.577	1.934
				k_{bm}			
0.2	0.046	0.134	0.199	0.142	-0.038	-0.742	-1.899
1/3	0.034	0.109	0.188	0.181	0.081	-0.392	-1.221
0.5	0.028	0.092	0.174	0.189	0.132	-0.207	-0.843
2	0.015	0.055	0.123	0.163	0.166	0.062	-0.182
3	0.013	0.050	0.112	0.153	0.161	0.084	-0.104
5	0.011	0.045	0.102	0.142	0.153	0.097	-0.044
				k_{bb}			
0.2	0.641	0.615	0.555	0.495	0.441	0.356	0.293
1/3	0.643	0.621	0.566	0.508	0.455	0.368	0.305
0.5	0.644	0.624	0.573	0.516	0.463	0.377	0.313
2	0.645	0.630	0.586	0.532	0.480	0.393	0.331
3	0.645	0.630	0.587	0.535	0.482	0.395	0.334
5	0.645	0.631	0.589	0.536	0.484	0.397	0.336
				k_{mb}			
0.2	0.011	0.033	0.064	0.086	0.100	0.115	0.122
1/3	0.008	0.026	0.054	0.074	0.089	0.104	0.111
0.5	0.006	0.022	0.047	0.066	0.079	0.095	0.102
2	0.003	0.013	0.030	0.044	0.055	0.067	0.073
3	0.003	0.011	0.027	0.040	0.050	0.061	0.066
5	0.003	0.010	0.024	0.037	0.045	0.054	0.058

Table 2. Stress Intensity Factor Ratios for a Saddle-Shaped Shell, R_1/R_2 = -0.5, a=10h, ν=0.3

$a/\sqrt{hR_2}$	0.1	0.25	0.50	0.75	1.0	1.5	2.0
k_{mm}	1.014	1.079	1.261	1.480	1.699	2.098	2.455
k_{bm}	-0.015	-0.045	-0.066	-0.039	0.025	0.195	0.335
k_{bb}	0.645	0.633	0.594	0.546	0.498	0.414	0.353
k_{mb}	-0.003	-0.011	-0.019	-0.020	-0.016	-0.004	0.006

Tables 1 and 2 show some of the calculated results for shells with positive or negative curvature ratio.

ACKNOWLEDGEMENT

This work was supported by NSF under the Grant ENG 7809737.

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APPENDIX A. Dimensionless Quantities

$$x = x_1/a \ , \ y = x_2/a \ , \ z = x_3/a \ . \tag{A.1}$$

$$u = u_1/a \ , \ v = u_2/a \ , \ w = u_3/a \ . \tag{A.2}$$

$$\beta_x = \beta_{11} \ , \ \beta_y = \beta_{22} \ , \ \phi = F/(a^2Eh) \ ,$$

$$\sigma_{xx} = \sigma_{11}/E \ , \ \sigma_{yy} = \sigma_{22}/E \ , \ \sigma_{xy} = \sigma_{12}/E \ . \tag{A.3}$$

$$N_{xx} = N_{11}/hE \ , \ N_{yy} = N_{22}/hE \ , \ N_{xy} = N_{12}/hE \ . \tag{A.4}$$

$$M_{xx} = M_{11}/h^2E \ , \ M_{yy} = M_{22}/h^2E \ , \ M_{xy} = M_{12}/h^2E. \tag{A.5}$$

$$V_x = V_1/hB \ , \ V_y = V_2/hB \ , \ B = \frac{5}{6}\frac{E}{2(1+\nu)} \ . \tag{A.6}$$

$$\lambda_1^4 = 12(1-\nu^2)a^4/h^2R_1^2 \ , \ \lambda_2^4 = 12(1-\nu^2)a^4/h^2R_2^2 \ ,$$

$$\lambda_{12}^4 = 12(1-\nu^2)a^4/h^2R_{12}^2 \ , \ \lambda^4 = 12(1-\nu^2)a^2/h^2 \ , \ \kappa = E/B\lambda^4 \ . \tag{A.7}$$

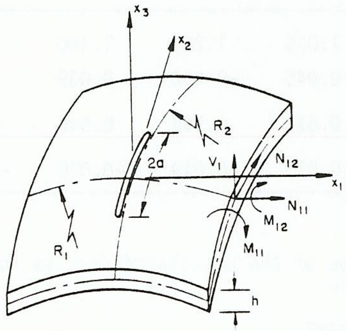

Fig. 1 Notation for the cracked shell.