VARIATIONAL BOUNDS AND QUALITATIVE METHODS IN FRACTURE MECHANICS

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INTRODUCTION

According to modern fracture mechanics, to determine the conditions of subsequent growth of a crack of given geometry, it is necessary to know the stress intensity factor in the points of the initial crack contour as well as in the points of all the subsequent positions of the crack contour. This is of minor importance in plane and axisymmetrical problems but gives rise to great difficulties in three-dimensional problems such as the problem of growth of an opening mode crack in the plane of symmetry of an elastic body.

The paper is concerned with some methods of determination of conditions sufficient for a crack to be dangerous or safe in the principal three-dimensional case mentioned above. The key to the problem is the notion of positivity which is introduced here. The problem is said to be positive if the application of arbitrary positive (wedging) tractions to crack surfaces gives rise to positive normal displacements of the surfaces and positive normal stresses in the plane of symmetry outside the crack. For positive problems there is the following comparison principle: the stress intensity factor at a given point of the crack contour grows as the crack extends outside some arbitrary small region around the point. The stress intensity factor grows also if the additional wedging forces are applied to the crack surfaces. It follows that for positive problems a given crack is more dangerous (i.e., gives rise to fracture during a shorter period of time) than any crack it contains and is less dangerous than any crack that contains it. This makes it possible to take into consideration the conditions of growth of cracks with comparatively simple ("standard") contours only.

DISCUSSION

The principle of comparison makes it possible to construct a two-sided estimate of the stress intensity factor at a given point on the arbitrary smooth contour of a crack considering contours of simpler form enveloping the given crack and enveloped by it and having a common tangent in the point under consideration.

It has been shown [1] that the problem of a plane crack in an infinite three-dimensional body is positive. In [1] it was demonstrated for the example of an elliptical crack in an infinite elastic body that the comparison principle is a highly efficient means of construction of two-sided estimates of stress-intensity factors. The theorem may be extended to include the case of an opening mode crack in a bounded body as soon as

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the boundary is sufficiently far from the crack. It has been proved that
the problem of an opening - mode crack situated in the central plane of an
elastic layer with traction-free faces is positive as soon as $d/h < 0.7$, $d$ - being the diameter of the plane domain occupied by the crack, $h$ - the
thickness of the layer. The crack opening and stress intensity factor at
any point of the crack contour diminish as the thickness of the layer grows,
the crack geometry and tractions on its surfaces being fixed.

The stress intensity factor is a local differential quantity, so that its
evaluation involves enormous difficulties. Nevertheless, the load distri-
bution being prescribed there exists a family of specific so-called "extremal" contours with the following two properties: 1) The stress in-
tensity factor is constant along the crack contour. 2) The elastic energy
of a body with a crack of given area attains its maximum value for cracks
with contours which belong to the family of extremal contours. The respec-
tive values of energy and stress intensity factor are functions of area
bounded by extremal contours, so the stress intensity factor may be expressed
through the derivative of the elastic energy on the crack area. The
extremal contours may be used as "barriers" for crack growth in the sense
of the introduction. As a result it becomes possible to express the notion
of a dangerous or a safe crack in terms of integral quantities such as the
crack area and elastic energy of the body with a crack. If the comparison
principle is valid for the body under consideration and there is an extremal
contour such that the corresponding stress intensity factor is equal
to its critical value, then all the cracks contained inside the contour
are safe and all the cracks containing the contour are dangerous. Consider
an extremal contour $\Gamma$ which bounds a domain $D$ of area $S$. It is assumed
that there is a supporting domain $G_0$ of area $S_0 < S$. Then the extremal
contour generally has two parts one of which $\Gamma_0 = \partial D_0$ and the second $\Gamma_0$
freely, i.e. lies outside $G_0$. The stress intensity factor on the free
part of $\Gamma$ is expressed through the corresponding values of elastic energy
$W$ as a function of area $S$: $W = W(S)$ by the formula

$$N^2/\sigma_{\Gamma}^2 = \frac{\mu}{(1-\nu)} \frac{dW}{dS}$$  

(1)

The true displacements in the points of the surface of a crack with a con-
tour of given form minimize the elastic energy $W$ of the cracked body. So
through the definition of an extremal contour is devised a solution of a
"maximum" problem:

$$N = \text{max} \min W$$

$G \in G_0$  

(2)

Consider an example of an extremal contour. Let a crack occupy a domain
$G_0$ in the plane $x_3 = 0$ in elastic space with surfaces acting normal tractions
acting on its surfaces:

$$\sigma_{13} = -p(1+\epsilon x_3^2), p = \text{const}, \epsilon = \text{const}$$

(3)

The constant $\epsilon$ is assumed to be small. If $\epsilon = 0$ the extremal contour is
evidently a circle containing $G_0$. For small $\epsilon$ and with symmetry assumed
it seems reasonable to seek the extremal contour as an elliptical contour of
form:

$$x_1 = a_1 \cos \theta, x_2 = b_1 \cos \theta, a_1 = a(1+\delta_1), b_1 = a(1+\delta_2), \delta_1, \delta_2 \ll 1$$

(4)

The unknown $\delta_1, \delta_2$ may be determined using the condition $\text{Neumann}$ along the
contour. Some algebra gives:

$$\delta_1 = \epsilon a^2 \frac{29}{345}, \delta_2 = \epsilon a^2 \frac{3}{115}$$

(5)

Now the equations (4), (5) give a family of extremal contours corresponding
to load of form (2), a being a parameter. The family may be used to esti-
mate the conditions of limiting equilibrium of plane cracks of arbitrary
geometry under loading of form (2) as outlined before.

To apply the approach presented here we must have some effective solutions
of three-dimensional crack problems for cracks bounded by plane contours.
Such solutions may be constructed by straightforward use of variational
and variational-difference methods. Some numerical results are presented.
The results may be obtained using medium range digital computers.

The conditions for a crack to be dangerous or safe having been expressed
in terms of an energy criterion there is a possibility of further simplifica-
tion of the elasticity problem under consideration. The simplification
is based on the following statement. Let the external loads $F_0 = \mathbf{F}(x)$ be
prescribed on part $S_0$ of the surface $S$ of an elastic body $D$, on the rest
of $S$ being prescribed the displacements, $u = \mathbf{u}(x)$. The elastic constants
of the material are considered to be functions of coordinates: $\lambda = \lambda(x), \mu = \mu(x)$. The quantity $Q$:

$$Q = \int_{S_0} \mathbf{F} \cdot \mathbf{u} + \mathbf{d} - \int_{S_0} \mathbf{F} \cdot \mathbf{u} + \mathbf{d}$$

(6)

may be considered as a functional of $\lambda$ and $\mu$; $Q = Q(\lambda, \mu)$. It may be shown
that the functional is monotonic: $Q'(x) > 0$ and $\mu' > \mu(x)$, $\mathbf{v} \times \mathbf{d}$, then $Q' = Q'(x) < 0$. In particular, for $\mathbf{d} = 0$ (the part $S_0$ of $S$ is clamped)
$Q$ is equal to the work done by the external loads. So it follows that the
work increases as the material rigidity in some subdomain of $S$ decreases
and vice versa. Thus it is possible to estimate strain energy for a given
cracked body in terms of the energies of bodies of simpler geometry with
cracks bounded by extremal (at the prescribed crack area) contours. Con-
sider, for example, infinite space with a crack under uniform tension
normal to the crack plane. Now the free extremal contours are, evidently,
circles; the elastic energy is equal to half the work done by the applied stress.
This assumes that the volume $V$ of a plane crack of arbitrary geometry and area $S$, under internal pressure $P$ is not greater than
the volume of a penny-shaped crack of the same area, so:

$$V \leq \frac{P}{E} \frac{16}{37} \frac{(1-\nu^2)}{S}$$

(7)

The inequality is an analogue of a well-known inequality for the capacity
of a plane domain $[2]$. In certain cases the energetic bounds may be applied directly to estimate
the stress intensity factors. For example, consider a circular crack of
radius $r$ around a spherical cavity of radius $p$. The crack and the cavity are
under an internal pressure $P$. The potential energy is, by using
linearity and dimensional arguments:

$$W = \frac{p^2 r^3}{2} \frac{\phi(r/p)}{2}$$

(8)
\( \phi \) being a dimensionless function. Through the statements of this section we have \( \phi' > 0 \). Thus:

\[
\frac{3W}{3\phi} = \frac{3p^2 E}{\mu} \phi\left( \frac{\phi}{\phi'} \right) \leq \frac{3p^2 E}{\mu} \phi\left( \frac{\phi}{\phi'} \right) = \frac{3W(\phi, \phi)}{\phi} \leq \frac{3W(\phi, \phi)}{\phi}
\]

(9)

It implies, using Irwin's formula:

\[
N^2 = N_0^2 = \frac{3p^2 E}{4\pi(1-\nu)}
\]

(10)

For \( \nu = 0.25 \), \( N \leq 0.56 \sqrt{\pi} \). For a penny-shaped crack of radius

\[
r_0, \quad N = \frac{\sqrt{r_0}}{\pi} \approx 0.45 \sqrt{r_0}.
\]

So a crack of radius \( \ell \) surrounding a spherical cavity of radius \( \rho \) is less dangerous than a penny-shaped crack of radius

\[
r_{\text{eff}} = \frac{\pi \ell}{8(1-\nu)} \quad \text{(for } \nu = 0.25, \ r_{\text{eff}} = \frac{\pi \ell}{3}).
\]

In elastic contact problems this makes it possible to construct bounds for the displacement of a die and/or the force acting on the die, with the complex geometries of contact areas and elastic body and various types of contact (sliding contact, frictional contact, etc.), on the basis of a solution of the respective problems for dies and/or bodies of simple geometries.

REFERENCES