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ON THE THREE DIMENSIONAL THEORIES OF CRACKED PLATES

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INTRODUCTION

It is well known since the beginning of Fracture Mechanics that the Strain-Energy-Release-Rate G_c depends on the thickness B of the specimen used; the value of G_c at smaller thicknesses could sometimes be seven times that for very thick specimens ((see for example Figure 3.36) in reference [1]). It is evident that the plasticity correction used after Irwin and based on plane-stress plane-strain argument could not help account for this variation and it was generally believed that a three-dimensional theory was needed to explain the triaxiality effect. A three-dimensional elasto-plastic theory is beyond the present reach. Even in the elastic domain, the problem seems to be extremely complicated. Nevertheless, recently two theories of elastic cracked plates have been proposed which reveal the variation of the stress-intensity factor K in the thickness direction. One would therefore be tempted to explain at least partially the thickness effect noting that G and K are related through material constants.

The aim of the present communication is:
1) to discuss two theories: one due to Hartranft and Sih [2] (we shall call it H-S theory) which starts with a certain proposed variation of stresses through the thickness, the other due to Folias using a certain integral representation for the displacements;
2) to present some numerical results obtained on a cracked plate of moderate thickness. These will be critically examined to evaluate the available theories;
3) finally, to look into some of the fundamental hypotheses of the present day fracture theory in the light of available 3-D fracture results.

THEORY PROPOSED BY HARTRANFT AND SIH

In a series of papers [2,3,4], Hartranft and Sih have developed an approximate 3-D theory of plates as applied to crack problems. In [3], they have shown that the singular part of the normal stress σ_{yy}, for the case of a plate subjected to uniform stress σ_o (Figure 1) can be written as

$$\sigma_{yy} = \frac{k_s(z)}{\sqrt{2r}} \cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2} \cdot \sin\frac{3\theta}{2}\right)$$

where the stress-intensity factor is

$$k_s(z) = g(p)\ \phi(1)\ \sigma_o\sqrt{a}\ f''(\zeta)$$

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where $\zeta = z/B$, $g(p)$ is a function of a constant parameter p and $\phi(1)$ is given as a function of the parameter p and the ratio B/a (Figure 2).

The mean value of $k_s(z)$

$$\bar{K}_s = \frac{1}{2B}\int_{-B}^{+B} k_s(z)\, dz = \frac{4}{p}\,\frac{\sin^2 p}{(2p+\sin 2p)}\,\phi(1)\,\sigma_o\sqrt{a}$$

which using p = 0,4 as suggested in [4] becomes

$$\bar{K}_s = \phi(1)\,\sigma_o\sqrt{a}$$

It is interesting to note that $\bar{K}_s$ varies with B/a through $\phi(1)$, a variation similar to the one implied by the experimental data. Thus one might expect to account for a partial thickness effect with the help of this theory.

THEORY PROPOSED BY FOLIAS [5]

Folias has treated the same problem using the method of Lure [6]. The analysis is sufficiently complex. The final result for the stress σ_{yy} in the crack-tip vicinity is

$$\sigma_{yy} = \Lambda\, F(\zeta)\,\frac{\sigma_o\sqrt{a}}{\sqrt{2r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\,.\,\sin\frac{3\theta}{2}\right),\quad 0 < \zeta < 1,$$

which leads to the stress-intensity factor

$$k_F(z) = \Lambda\, F(\zeta)\,\sigma_o\sqrt{a},\quad F(\zeta) = \frac{1}{2}\left\{\frac{1}{(1-\zeta)^{2\nu}} + \frac{1}{(1+\zeta)^{2\nu}}\right\}$$

and $\Lambda = \Lambda(B/a)$ is shown in Figure 3 for $\nu = 1/3$.

The average value of $k_F(z)$ is given by

$$\bar{K}_F = \frac{\Lambda\,\sigma_o\sqrt{a}}{(1-2\nu)(2)^{2\nu}}$$

It should be noted that the mean value of K_F is greater than the two-dimensional value except for $\nu = 0$ when the plane-stress result is recovered. Moreover, one finds that contrary to the results of H-S theory, Folias' theory indicates an increase of stresses as z/B is varied from 0 to the value of 1. On the other hand, the variation in the mean value of $\bar{K}_F$ ($\nu \neq 0$) with respect to B/a (Figure 3) is of the order of 11% and one would tend to believe that a 3-D elastic theory would not give even a partial answer to the experimental variation in G_c with respect to thickness.

NUMERICAL RESULTS [7]

We have carried out certain numerical computations for the value of K in a moderately thick centrally through the thickness cracked plate.

The configuration used is that of Figure 1 with a width of 60 mm and two values for the thickness and for the crack length to obtain finally three different values of B/a (1.0, 0.75 and 0.25). The computations were performed using two different methods:

The Finite Element Method

We shall denote

$$K_\sigma = \lim_{r\to 0} \sqrt{2r}\;\sigma_{yy} \quad \text{(using stress data)}$$

$$K_u = \lim_{r\to 0} \frac{E\,U_y}{2\sqrt{2r}\,(1-\nu^2)} \quad \text{(using displacement data and plane strain assumption)}$$

Note that, although both K_σ and K_u values are given in the text, K_u is the solution to be compared because the computer codes we used assure the displacement compatibility as most of the F.E. computer programmes. This point has already been stressed in the literature [8] and we shall not go into more details:

The F.E. programmes used are:
1) SAFE 2D for a two dimensional analysis in order to compare with the three-dimensional analysis
2) SAP IV for the analysis of the finitely thick plate. In this case 16-node brick elements were used. The mesh is shown in Figure 4 for 1/8th of the plate considered. We have used 336 elements. Two values of B/a = 0.75 (Case I) and B/a = 0.25 (Case II) were run with a uniform tensile stress of 147 MPa applied at the edges of the plate parallel to the crack.

The results concerning σ_{yy}- distribution through the thickness for the plane y = 0 are shown in Figure 5. The computed mean K-values are given in Table 1. These results will be discussed later.

The Boundary Integral Equation Method

This method has been developed due to the efforts of Cruse [9], Lachat and Watson [10] and others. The results we present here were obtained with a EITD programme developed at CETIM, using a surface discretisation shown on Figure 6. This programme gives the value of the J-integral for any given closed contour around the crack-tip. We calculated the J-value using three contours in order to verify that this value is indeed independent of the contour. The K_J value is

$$K_J = \left[\frac{E\,J}{\pi(1-\nu^2)}\right]^{1/2} \quad \text{(cf. Table 1)}$$

Note that the stress distribution obtained with the EITD programme was practically identical to the one calculated with SAP programme.

Before comparing the different K values, we would like to discuss the precision on numerical results. Firstly, before carrying out the costly 3-D analysis, we check that the mesh used was sufficiently refined comparing 2-D calculations with available theoretical value. ($K_{Th} = \alpha\,\sigma\sqrt{a}$, α being the finite-width-correction factor (11)). The comparison was extremely good (cf. Table 1). After this step, we proceed to 3-D calculations. The use of two different methods (F.E. and B.I.E.M.) gives us another cross-check on 3-D results. The comparison was once again extremely good (cf. Table 1). This confirms our confidence in the results and we feel that their precision is certainly better than 10%.

COMPARISON BETWEEN THEORETICAL AND NUMERICAL RESULTS

Stress Distribution

From Figure 5 we note that the numerically calculated stress σ_{yy} falls down as we go from the centre of the plate towards the free surfaces. This result is in contradiction to Folias' theory. The H-S theory, though shows the same tendencies as the numerical results, predicts that the stress distribution for different values of B/a should be sufficiently different. For example, the ratio of stress in cases I and II (same crack-length), in the vicinity of crack-tip

$$\left(\sigma_{yy}\right)_{I} / \left(\sigma_{yy}\right)_{II} = \left[\phi(1)\right]_{I} / \left[\phi(1)\right]_{II} = 1.42$$

instead of nearly 1 as given by the numerical analysis.

K-Values

From Table 1, for the 3-D geometry, we shall make two remarks on the mean value of K:

- firstly, concerning the values of K: we find that the 3-D numerical results ($\overline{K}_u$) are close to 2-D values while the predictions of the H-S theory are very much lower and that of Folias very much higher.
- secondly, concerning the variation in K-values with respect to thickness: The numerical results indicate that there is hardly any effect of thickness on the K-values. This result is similar to that given by Folias' theory whereas the H-S theory predicts a variation of 42% in going from Case I to Case II.

It might be of interest to point out here that the analysis of Sternberg and Sadowsky [12], though for the case of a circular hole in a plate of arbitrary thickness, also showed little dependence of the stress distribution on the thickness of the plate.

Figure 7, indicating the distribution of the S.I.F. along the crack front for Cases I and II, was drawn to visualize still better the comparison between different methods. Note that the H-S theory predicts that for moderately thick plates, the results are quite different from the plane solutions, which does not seem to be the case. The Folias' theory though predicts that the thickness effect is about 10%, gives a K-distribution along the crack front through the thickness which is entirely different from that of numerical results. Moreover, the mean value of K from the Folias' theory ($\overline{K}_F$), even for large thicknesses is much higher than the theoretical 2-D plane-strain solution.

CONCLUSION

It is evident that the three dimensional problem of cracked plates is not yet fully resolved even in the linear elastic domain. But the important conclusion to which the present study leads us is that one would be incapable to predict the experimentally observed variation of G_c with respect to thickness, even through an exact 3-D elastic theory. This comes from the fact that the 3-D numerical results are close to the 2-D ones.

On the other hand, looking at the parameter G_c as obtained through the Griffith-Irwin theory, we find that its value remains nearly the same for the cases of plane-stress and plane-strain, although the plastic flow at the crack tip is entirely different in both the cases. This leads us to believe that certain hypothesis of this theory in formulating G_c may have to be re-examined.

In particular, the plastic energy dissipation rate, considered constant in this theory was shown to vary with the geometry and the applied loading (see [13] and the references given there). This result might be of significance in formulating a realistic fracture governing parameter.

REFERENCES

1. TETELMAN, A. S. and McEVILY, A. J., "Fracture of Structural Materials", John Wiley, New York, 1967, 137.
2. HARTRANFT, R. J. and SIH, G. C., Int. J. Engng. Sci., 8, 1970, 711.
3. SIH, G. C. and HARTRANFT, R. J., Int. J. Fracture, 9, 1973, 75.
4. VILLARREAL, G., SIH, G. C. and HARTRANFT, R. J., ASME Paper No. 75-APMW-29, 1975.
5. FOLIAS, E. S., ASME Paper No. 75-APM-24, 1975.
6. LURE, A. I., "Three Dimensional Problems of the Theory of Elasticity", Interscience Publishers, New York, 1964.
7. BHANDARI, S. K. and BARRACHIN, B., unpublished G.A.A.A. Report MP.D310, May 1976.
8. TRACEY, D. M., Nuclear Engng. and Design, 26, 1974.
9. CRUSE, T. A., Computers and Structures, 3, 1973, 509.
10. LACHAT, J. C. and WATSON, J. O., App. Mech. Div. Nat. Conf., A.S.M.E., New York, 1975.
11. BHANDARI, S. K., GATEAU, M. and SERTOUR, G., Int. J. Fracture 12, 1976, 533.
12. STERNBERG, E. and SADOWSKY, M. A., J. App. Mech. 1949, 27.
13. BHANDARI, S. K., Int. J. of Fracture, 9, 1973, 345.

Table 1 Mean Value of S.I.F. as Given by Different Methods (MPa. $\sqrt{m}$)

			CASE I	CASE II	CASE III
CRACK LENGTH (a,mm)			10.0	10.0	7.5
RATIO B/a			0.75	0.25	1.0
2 - D VALUES	FINITE ELEMENTS	(K_σ)	18.2		
		K_u	15.65		
	THEORY*	K_{Th}	15.55		13.11
3 - D VALUES	FINITE ELEMENTS	$(\overline{K}_\sigma)$	18.44	18.35	
		$\overline{K}_u$	15.52	15.52	
	INTEGRAL EQUATIONS	K_J	15.53	16.2	12.83
	H-S THEORY	$\overline{K}_s$	13.21	9.27	12.11
	FOLIAS THEORY	$\overline{K}_F$	23.64	23.13	18.78

$^*K_{Th} = \alpha\, \sigma\sqrt{a}$

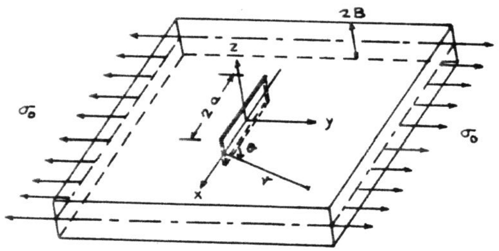

Figure 1 Plate with a Central Crack

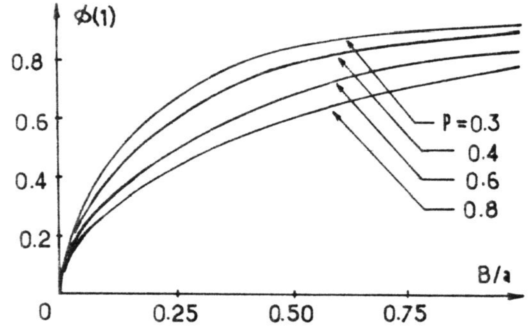

Figure 2 ϕ (1) as a Function of B/a and p (from reference [1])

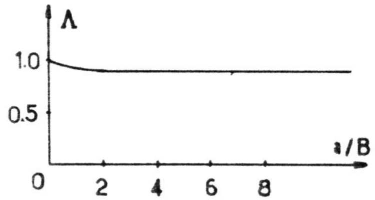

Figure 3 Λ as a Function of a/B (from reference [5])

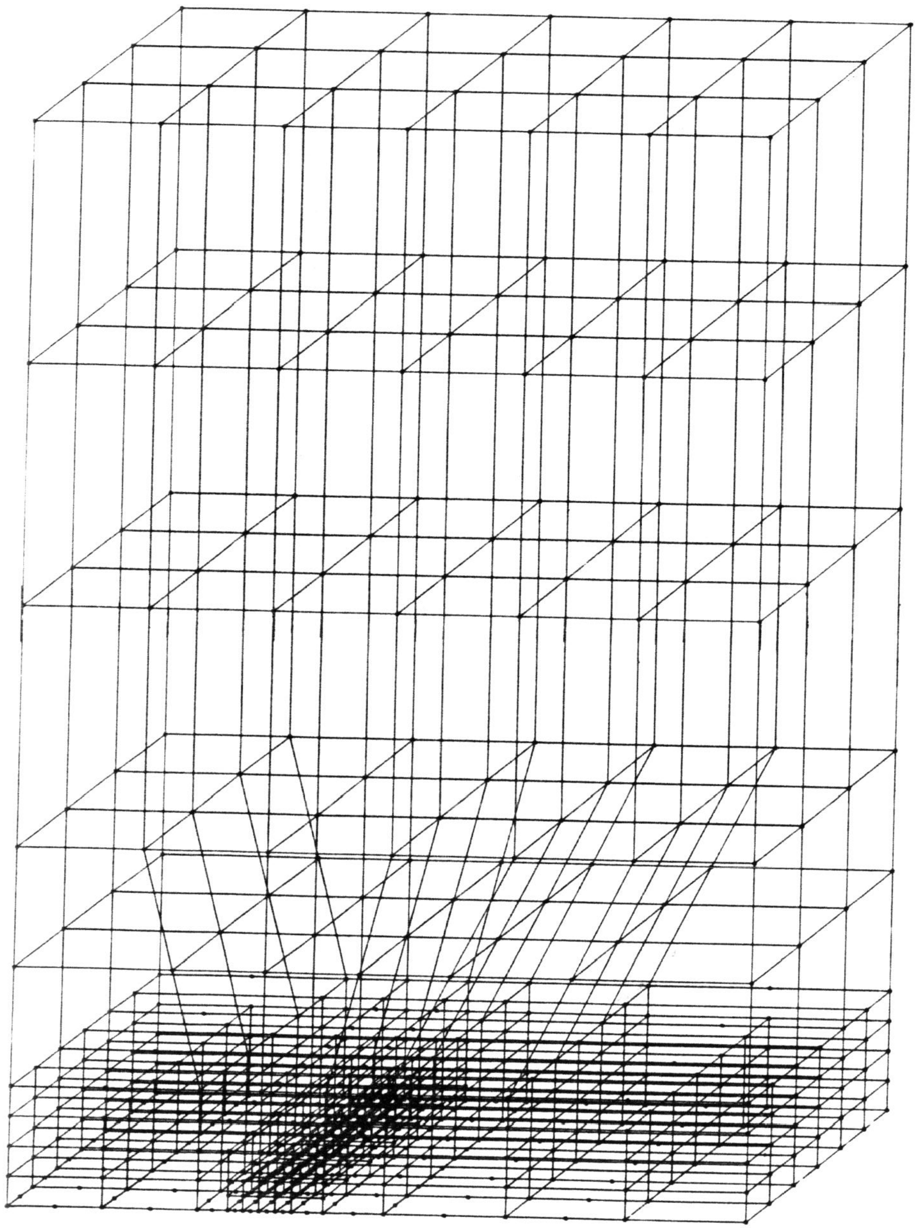

Figure 4

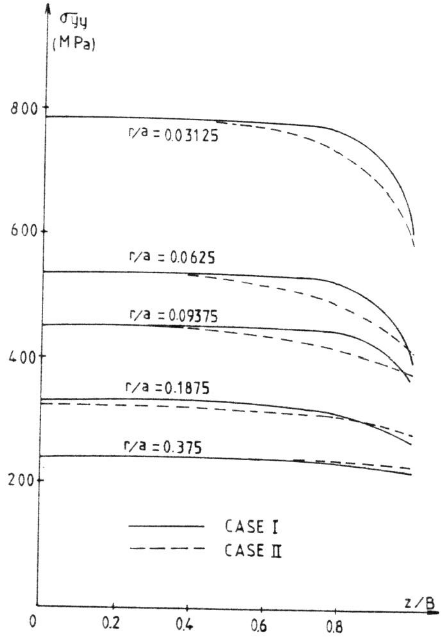

Figure 5 Normal Stress Distribution Through the Thickness of the Plate in the Plane y = 0

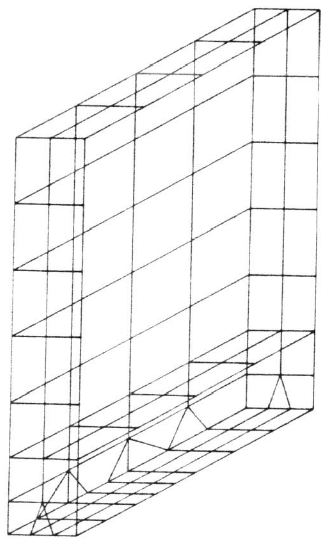

Figure 6 Surface Discretisation for Integral-Equations Analysis

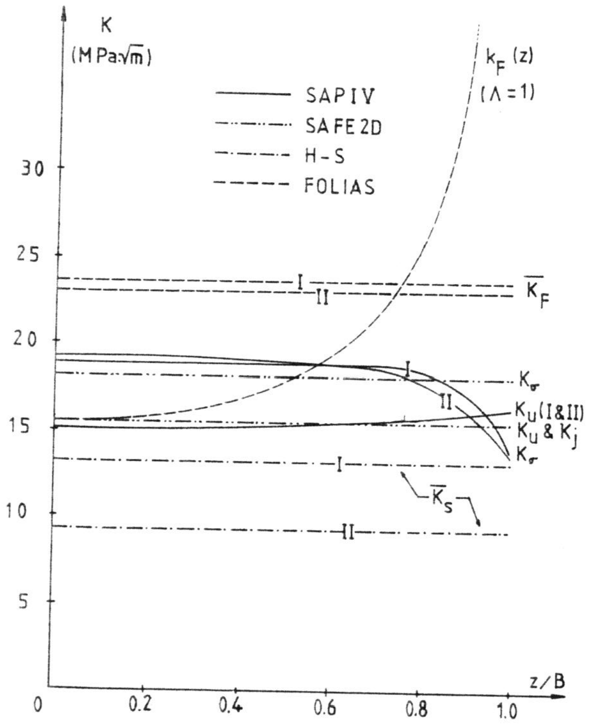

Figure 7 Distribution of Stress-Intensity Factors for Cases I and II