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ANALYTICAL DETERMINATION OF STRESS INTENSITY FACTORS OF ECCENTRIC CRACKS BY VARIATIONAL METHOD

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INTRODUCTION

A direct application of variational methods to the analysis of a crack has not been found, except for analysis by finite element methods. A series of researches [1 - 4] has shown that a variational method can serve as a useful tool in the following cases:

(1) Analyses of a cracked finite plate with good accuracy and rapid convergence [1, 2].
(2) Analyses of a cracked plate with complicated or unknown boundary conconditions, being combined with experimental stress analysis techniques [3, 4].

Previously the authors proposed an analytical method for determination of the stress intensity factors of a crack in a finite plate by a variational method [1, 2]. In these papers an analytical solution for a crack in an infinite plate was applied to the analysis of a crack in a finite plate. In the previous papers [1, 2], a rectangular plate with an edge crack was analysed by means of a stress function for a semi-infinite crack [5, 6] and the principle of minimum potential energy as a variational principle; accurate numerical results were obtained. However, there is a possibility that some reduction of accuracy might occur when the method is applied to such cases as internally cracked plates or mixed boundary value problems; some improvements or developments of the method may therefore be required.

In this paper, the formulation of the variational principle is improved and extended into a general form more convenient for the analysis of various crack problems. To examine the accuracy of results and the applicability of the present method, two types of rectangular plates with an internal and eccentric crack, that is, a plate pulled by uniaxial uniform tension and a plate pulled by rigid clamped ends (i.e. uniform displacement at the ends), are analysed. In the former case, the numerical results are directly compared with the results obtained by a collocation technique. The latter is analysed for the purpose of clarifying the following phenomenon with regard to the behaviour of an eccentric crack, which has become of general interest and has remained unsolved. The phenomenon is that a slightly eccentric crack in a plate grows in such a fashion that this eccentricity is decreased. In this paper, it is found that the phenomenon is well explained by the concept of stress intensity factors.

By two numerical examples described above, the availability of the new method proposed in this paper is thought to be assured.

Moreover, the new method can effectively be applied to the edge crack problems analysed in the previous paper [1].

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FORMULATION OF VARIATIONAL PRINCIPLE

Consider a mixed boundary value problem with prescribed boundary traction $\overline{T}_i$ over the boundary Γ_σ, and prescribed displacement $\overline{u}_i$ over the boundary Γ_u. For effective application of the variational principle to the analysis of mixed boundary value problems, the Hellinger-Reissner formulation [7, 8] is chosen. The functional Π_R of the Hellinger-Reissner formulation is given by

$$\Pi_R = \int_S \left[\frac{1}{2}\sigma_{ij}(u_{i,j}+u_{j,i})-B(\sigma_{ij})-\overline{F}_i u_i\right]dS-\int_{\Gamma_\sigma}\overline{T}_i u_i d\Gamma$$
$$-\int_{\Gamma_u} T_i(u_i-\overline{u}_i)d\Gamma \tag{1}$$

where σ_{ij} is stress; u_i is displacement; $B(\sigma_{ij})$ is the complementary energy function expressed in terms of the stress; $\overline{F}_i$ is prescribed body force; S is the area of the cracked plate; and Γ is the outer boundary of S, composed of Γ_σ and Γ_u; T_i is the traction force. Equation (2) is the definition of the traction force.

$$T_i = \sigma_{ij}n_j \tag{2}$$

where n_j is the direction cosine of the unit normal drawn outwards on Γ. The Euler equations for equation (1) are

$$\frac{1}{2}(u_{i,j}+u_{j,i}) = S_{ijkl}\sigma_{kl} \tag{3}$$

$$\sigma_{ij,j}+\overline{F}_i = 0 \tag{4}$$

where S_{ijkl} are the compliance coefficients. In equation (1), σ_{ij}, T_i and u_i can be assumed to be independent of each other. However, in the case of crack problems, they have to be chosen so that the Euler equations (equations (3) and (4)) are exactly satisfied, by means of the analytical solution which satisfies the stress free condition along the crack surfaces.

In this paper, the body forces $\overline{F}_i$ in equation (1) are assumed equal to zero. Applying Gauss' theorem to equation (1) and substituting equation (4) into it,

$$\Pi_R = \frac{1}{2}\int_{\Gamma_\sigma} T_i u_i d\Gamma - \frac{1}{2}\int_{\Gamma_u} T_i u_i d\Gamma - \int_{\Gamma_\sigma} u_i \overline{T}_i d\Gamma + \int_{\Gamma_u} T_i \overline{u}_i d\Gamma \tag{5}$$

or, in matrix form,

$$\Pi_R = \frac{1}{2}\int_{\Gamma_\sigma} \underline{T}^T\underline{u}d\Gamma - \frac{1}{2}\int_{\Gamma_u} \underline{T}^T\underline{u}d\Gamma - \int_{\Gamma_\sigma} \underline{u}^T\overline{\underline{T}}d\Gamma + \int_{\Gamma_u} \underline{T}^T\overline{\underline{u}}d\Gamma \tag{6}$$

where the superscript T is the transpose of a matrix or a vector. It is noted that this modified form of the Hellinger-Reissner formulation can be

conveniently employed for various classes of crack problems, since one can avoid the area integral in S that includes the region of the high gradient of stress near a crack tip, thus avoiding the problems of accuracy associated with numerical integration in such regions. Instead, it is sufficient to carry out the line integrals on Γ using equation (5) or (6), which is expected to be comparatively simple and accurate.

PROCEDURES OF ANALYSIS

To illustrate the method, the results of analytical calculation for two types of internally and eccentrically cracked rectangular plates will be shown. One is pulled under uniform tension (shown in Figures 1 and 2) and the other is pulled under uniform displacement (at its clamped ends) (shown in Figure 3).

Since the former problem has been solved by Terada and Isida [9] with the collocation method, the accuracy of the results obtained by the variational method can be examined by comparison with their results.

Analytical Solution for an Internally Cracked Plate

In the theory of two-dimensional isotropic elasticity, the stresses (σ_x, σ_y, τ_{xy}) and the displacements (u_x, u_y) are generally expressed in terms of two analytical functions $\phi(z)$ and $\Omega(z)$ of the variable $z = x + iy$. They are

$$\sigma_x+\sigma_y = 2[\phi(z)+\overline{\phi(z)}] \tag{7}$$

$$-\sigma_x+\sigma_y+2i\tau_{xy} = 2[(\overline{z}-z)\phi'(z)-\phi(z)+\overline{\Omega}(z)] \tag{8}$$

$$2\mu(u_x+iu_y) = \eta\int\phi(z)dz-\int\Omega(\overline{z})d\overline{z}+(\overline{z}-z)\overline{\phi(z)} \tag{9}$$

where i is the imaginary number; $\eta = (3-\nu)/(1+\nu)$ for plane stress and $\eta = 3-4\nu$ for plane strain; $\mu = E/2(1+\nu)$; E and ν are Young's modulus and Poisson's ratio, respectively; $\overline{(\ )}$ or $(\ )'$ denotes conjugation or differentiation, respectively, with respect to z.

When $\phi(z)$ and $\Omega(z)$ satisfy the conditions of traction free crack surfaces, they are expressed [10, 11] as

$$\phi(z) = \frac{\sum_{n=0}^{N} A_n z^n}{\sqrt{z^2-a^2}} + \sum_{n=0}^{N} B_n z^n \qquad \Omega(z) = \frac{\sum_{n=0}^{N} A_n z^n}{\sqrt{z^2-a^2}} - \sum_{n=0}^{N} B_n z^n \tag{10}$$

where A_n and B_n are unknown complex coefficients; N is a finite integer; 2a is crack length; and the coordinates are shown in Figures 1, 2 and 3.

Moreover, $\phi(z)$ and $\Omega(z)$ have to satisfy the conditions of single-valuedness of the displacements, that is, the following equation has to be satisfied for an arbitrary closed curve L around the crack.

$$\eta\int_L\phi(z)dz-\int_L\Omega(\overline{z})d\overline{z} = 0 \tag{11}$$

Application of Variational Principle

Rectangular plates with an eccentric crack, as shown in Figures 1, 2 and 3, are analysed. On account of the symmetry of the problem with respect to the x-axis, the unknown complex coefficients (A_n and B_n) in equation (10) have the property that A_n and B_n $(n = 0,1,2,\ldots,N)$ are real. From the condition of single-valuedness of the displacements, equation (11), the following relation between unknown coefficients is obtained.

$$A_o + \sum_{n=1}^{N} A_{2N}\, M(2n) = 0 \tag{12}$$

where $M(2n) = (2n-1)a^2 M(2n-2)/(2n)$

$M(0) = 1$

Then, from equations (2), (7), (8), (9), (10), and (12), the traction forces $\underline{T}$ and displacements $\underline{u}$ are given in a matrix form by

$$\underline{T} = \begin{Bmatrix} T_x \\ T_y \end{Bmatrix} = \underline{R_\alpha}\, \underline{\alpha} \tag{13}$$

$$\underline{u} = \begin{Bmatrix} u_x \\ u_y \end{Bmatrix} = \underline{U_\alpha}\, \underline{\alpha} \tag{14}$$

where

$$\underline{\alpha}^T = [A_1 \ldots A_N B_1 \ldots B_N] \tag{15}$$

$$\underline{R_\alpha} = \begin{bmatrix} g_x^1 \ldots g_x^N g_x^{N+1} \ldots g_x^{2N} \\ g_y^1 \ldots g_y^N g_y^{N+1} \ldots g_y^{2N} \end{bmatrix} \tag{16}$$

$$\underline{U_\alpha} = \begin{bmatrix} h_x^1 \ldots h_x^N h_x^{N+1} \ldots h_x^{2N} \\ h_y^1 \ldots h_y^N h_y^{N+1} \ldots h_y^{2N} \end{bmatrix} \tag{17}$$

and all of g_x^m, g_y^m, h_x^m and h_y^m $(m = 1,2,\ldots,2N)$ are the functions of z.

[Example 1] "A rectangular plate with an eccentric crack under uniform tension"

In this case, the terms including Γ_u vanish in equation (5) or (6). Furthermore, only the case for which the body force $\overline{F}$ is equal to zero is addressed. Then, substituting equations (13) and (14) into equation (5) or (6),

$$\Pi_R = \frac{1}{2}\underline{\alpha}^T \underline{H}\, \underline{\alpha} - \underline{\alpha}^T \underline{G} \tag{19}$$

where

$$\left.\begin{aligned} \underline{H} &= \frac{1}{2}\int_\Gamma (\underline{R_\alpha^T}\, \underline{U_\alpha} + \underline{U_\alpha^T}\, \underline{R_\alpha})\, d\Gamma \\ \underline{G} &= \int_{\Gamma_\sigma} \underline{U_\alpha^T}\, \underline{\overline{T}}\, d\Gamma \end{aligned}\right\} \tag{20}$$

From the stationary condition of equation (19) with respect to $\underline{\alpha}$, we have

$$\underline{\alpha} = \underline{H}^{-1}\, \underline{G} \tag{21}$$

Then, from equation (21), all of coefficients in equation (15) are obtained.

When the coordinates are given as shown in Figures 1, 2 and 3, stress intensity factors are defined by

$$K_j = K_{Ij} - iK_{IIj} = 2\sqrt{2\Pi}\, \lim_{z\to z_j}\left[\sqrt{z-z_j}\; \phi(z)\right] \quad (z_j = \pm a) \tag{22}$$

Therefore, stress intensity factors are given by equations (21) and (22). Figures 1 and 2 show the numerical results of dimensionless stress intensity factors, F_{IA} and F_{IB}, at the tips A and B of the crack, respectively. These present results by the variational method coincide with the results [9] by a collocation technique up to three or four figures, as shown in Table 1.

Next, to check the accuracy, the variations of F_{IA} and F_{IB} with increase of the number of terms 2N in equation (10) are examined. Table 1 shows a typical example for the cracked plate with $e/w = 0.3$ and $\lambda = 0.3$ (see Figure 1). The numerical convergence is quite excellent and the errors are less than one per cent when 2N is more than 6.

[Example 2] "A rectangular plate with an eccentric crack under uniform displacement (clamped ends)"

Also, in this case, the body forces $\overline{F}$ are assumed equal to zero. Substituting equations (13) and (14) into equation (5) or (6), we have

$$\Pi_R = \frac{1}{2}\underline{\alpha}^T \underline{H_\sigma}\, \underline{\alpha} - \frac{1}{2}\underline{\alpha}^T \underline{H_u}\, \underline{\alpha} - \underline{\alpha}^T \underline{G_\sigma} + \underline{\alpha}^T \underline{G_u} \tag{23}$$

where

$$\left.\begin{aligned} \underline{H_\sigma} &= \frac{1}{2}\int_{\Gamma_\sigma} (\underline{R_\alpha^T}\, \underline{U_\alpha} + \underline{U_\alpha^T}\, \underline{R_\alpha})\, d\Gamma \\ \underline{H_u} &= \frac{1}{2}\int_{\Gamma_u} (\underline{R_\alpha^T}\, \underline{U_\alpha} + \underline{U_\alpha^T}\, \underline{R_\alpha})\, d\Gamma \\ \underline{G_\sigma} &= \int_{\Gamma_\sigma} \underline{U_\alpha^T}\, \underline{\overline{T}}\, d\Gamma \\ \underline{G_u} &= \int_{\Gamma_u} \underline{R_\alpha^T}\, \underline{\overline{U}}\, d\Gamma \end{aligned}\right\} \tag{24}$$

From the stationary condition of equation (23) with respect to $\underline{\alpha}$,

$$\underline{\alpha} = (\underline{H_\sigma} - \underline{H_u})^{-1} (\underline{G_\sigma} - \underline{G_u}) \tag{25}$$

And then, stress intensity factors are obtained from equations (22) and (25).

Numerical results are shown in Figure 3. It is found that under the appropriate condition of crack length and eccentricity of the crack, the stress intensity factor K_{IB} at crack tip B (with smaller distance from the centre of this plate) can be slightly greater than the stress intensity factor K_{IA} at crack tip A.

The uniform displacement condition given to an asymmetrically cracked plate causes a negative in-plane bending moment which acts so that the crack tip A closes. An unsolved phenomenon, that a slightly eccentric crack in a plate grows such that the eccentricity decreases, can be explained by a difference of the stress intensity factors. A similar argument can probably be applied to a double edge cracked plate with clamped ends or a pin-loaded eccentric plate.

CONCLUDING REMARKS

Based on a variational principle, a new analytical method for determination of the stress intensity factors of a crack in a finite plate is proposed. By means of a modified Hellinger-Reissner formulation as presented above, mixed boundary crack problems can be solved. The numerical results indicate that by the present method an accurate evaluation of the stress intensity factors can easily be done with rapid convergency.

A phenomenon, that a slightly eccentric crack in a plate pulled by cyclic loads at the clamped ends grows so that the eccentricity decreases, is well explained by the stress intensity factors obtained by the present method.

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Table 1 Convergence and Comparison of Dimensionless Stress Intensity Factors of an Eccentric Crack in a Square Plate under Uniform Tension

$e/w=0.3$, $\lambda=0.3$

2N	F_{IA}	F_{IB}
6	1.056	1.054
8	1.059	1.055
12	1.060	1.057
16	1.063	1.060
20	1.064	1.061
24	1.065	1.061
26	1.065	1.062
28	1.065	1.062
30	1.065	1.062
32	1.065	1.062
collocation method*	1.066	—

*reference [9]

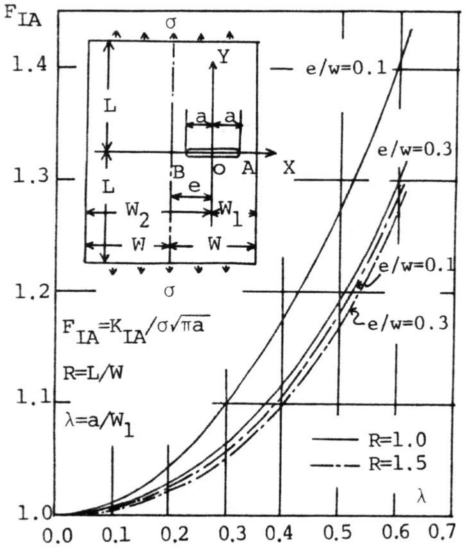

Figure 1 Dimensionless Stress Intensity Factor (F_{IA}) of the Tip (A) of an Eccentric Crack in a Plate Under Uniform Tension

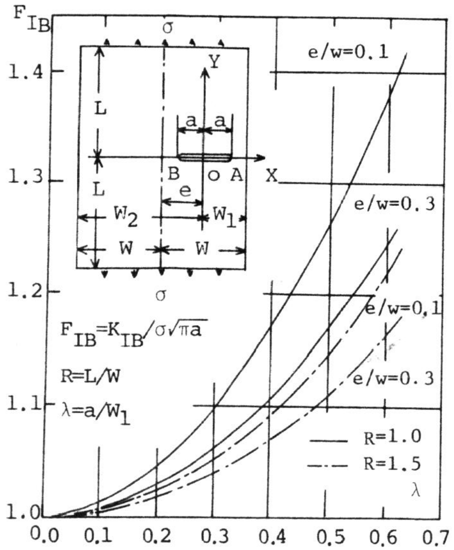

Figure 2 Dimensionless Stress Intensity Factor (F_{IB}) of the Tip (B) of an Eccentric Crack in a Plate Under Uniform Tension

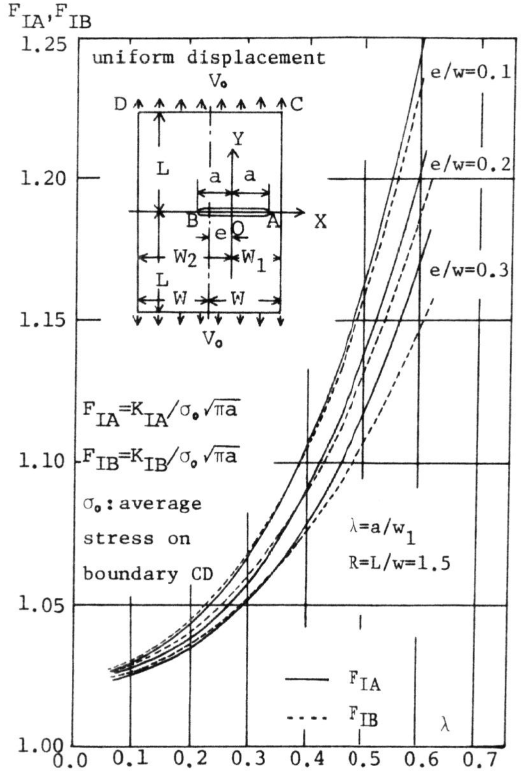

Figure 3 Dimensionless Stress Intensity Factors (F_{IA} and F_{IB}) of an Eccentric Crack in a Plate Under Uniform Displacement