AN EQUIVALENT INCLUSION METHOD FOR A THREE-DIMENSIONAL LENS-SHAPED CRACK IN ANISOTROPIC MEDIA

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INTRODUCTION

Micro cracks in materials sometimes take a three-dimensional lens-shape. In this paper crack opening displacements, crack extension forces, stress concentration factors for $a_3 \neq 0$ and stress intensity factors for $a_3 = 0$ are developed through the use of the equivalent inclusion method for an isolated three-dimensional lens-shaped crack under simple tension and pure shear, where a_3 is the smallest principal axis of the ellipsoid.

CRACK OPENING DISPLACEMENT

A three-dimensional lens-shaped crack is given by Figure 1 or by

$$\Omega:\ x_1^2/a_1^2 + x_2^2/a_2^2 + x_3^2/a_3^2 \leq 1 \tag{1}$$

where a_3 is smaller in comparison to a_1, a_2. The elastic constants of domain Ω are zero. An applied stress (σ_{ij}^0 at infinity) becomes $\sigma_{ij}^0 + \sigma_{ij}$ in the neighbourhood of Ω. The stress disturbance σ_{ij} is equivalent to the stress caused by eigenstrains ε_{ij}^* (phase transformation strains) defined in Ω, assuming the elastic constants of Ω to be the same as those of the matrix (denoted by $C_{ijk\ell}^0$). ε_{ij}^* are determined from

$$\sigma_{ij}^0 + \sigma_{ij} = 0,\ \sigma_{ij} = C_{ijk\ell}^0\left(u_{k,\ell} - \varepsilon_{k\ell}^*\right) \text{ in } \Omega, \tag{2}$$

where u_i is the displacement field due to ε_{ij}^*. We found by Green's function technique [1] that when ε_{ij}^* are constant and $a_3 \ll a_1$, a_2,

$$u_{i,k} \approx (1/4\pi)C_{j\ell mn}^0 \varepsilon_{mn}^* \left[4\pi G_{ijk\ell}(0,0,1) - a_3\Pi_{ijk\ell}\right] \tag{3}$$

where

$$G_{ijk\ell}(\bar\xi_1,\bar\xi_2,\bar\xi_3) = N_{ij}(\bar{\underset{\sim}{\xi}})\bar\xi_k\bar\xi_\ell/D(\bar{\underset{\sim}{\xi}})$$

$$\Pi_{ijk\ell} = \int_{S^2} \frac{a_1a_2}{(a_1^2\cos^2\theta + a_2^2\sin^2\theta)^{3/2}} \frac{\bar\xi_3}{(1-\bar\xi_3^2)^{1/2}} \frac{\partial}{\partial\bar\xi_3} G_{ijk\ell}(\bar{\underset{\sim}{\xi}})dS(\bar{\underset{\sim}{\xi}}) \tag{4}$$

$$\bar\xi_1 = (1-\bar\xi_3^2)^{1/2}\cos\theta,\ \bar\xi_2 = (1-\bar\xi_3^2)^{1/2}\sin\theta,\ dS(\bar{\underset{\sim}{\xi}}) = d\theta d\bar\xi_3.$$

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$N_{ij}(\xi)$, $D(\xi)$ are the cofactor and the determinant of matrix $(C^0_{ipjq}\bar{\xi}_p\bar{\xi}_q)$ respectively and S^2 is the unit sphere $\bar{\xi}_i\bar{\xi}_i = 1$. Equation (3) is linear with respect to a_3.

When $\sigma^0_{ij} = \sigma^0_{33}$ (simple tension) and the crystalline directions are parallel to the principal axes direction of Ω, equation (2) gives a non-zero component of ε^*_{ij} as

$$a_3\varepsilon^*_{33} = 4\pi\sigma^0_{33}/C^0_{33mn}C^0_{pq33}\Pi_{mpnq} \tag{5}$$

When $\sigma^0_{ij} = \sigma^0_{31}$ (pure shear) and the crystalline directions are parallel to the principal axes directions of Ω, a non-zero component of ε^*_{ij} is obtained as

$$a_3\varepsilon^*_{31} = 2\pi\sigma^0_{31}/C^0_{31mn}C^0_{pq31}\Pi_{mpnq}. \tag{6}$$

From the dislocation theory [2], the crack opening displacements u_3 for (5) and u_1 for (6) (displacements on Ω) are given by $\varepsilon^*_{ij}h$, where h is the half thickness of Ω in the x_3 direction:

$$h = a_3(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2}. \tag{7}$$

Since the right hand sides in (5) and (6) do not contain a_3, $\varepsilon^*_{ij} \to \infty$ for $a_3 \to 0$.

CRACK EXTENSION FORCES

The interaction energy between σ^0_{ij} and the crack is given by

$$\Delta U = -\frac{2}{3}\pi a_1 a_2 a_3 \varepsilon^*_{ij}\sigma^0_{ij}\ . \tag{8}$$

When the crack is expanding in the x_2 direction, keeping a_1, a_3 constant, the crack extension force G is given by $G = -\partial(\Delta U)/\partial a_2$. For the simple tension it becomes

$$G = 8\pi^2 a_1(\sigma^0_{33})^2(f/g)/C^0_{33mn}C^0_{pq33}\Pi_{mpnq} \tag{9}$$

where

$$f = C^0_{33ik}C^0_{j\ell 33}\int_{S^2}\frac{\xi_3}{(1-\bar{\xi}_3^2)^{1/2}}\left\{\frac{\partial}{\partial\bar{\xi}_3}G_{ijk\ell}(\xi)\right\}\frac{a_2^2\sin^2\theta\ ds(\xi)}{(a_1^2\cos^2\theta+a_2^2\sin^2\theta)^{5/2}} \tag{10}$$

$$g = C^0_{33ik}C^0_{j\ell 33}\int_{S^2}\frac{\xi_3}{(1-\bar{\xi}_2^2)^{1/2}}\left\{\frac{\partial}{\partial\bar{\xi}_3}G_{ijk\ell}(\xi)\right\}\frac{ds(\xi)}{(a_1^2\cos^2\theta+a_2^2\sin^2\theta)^{3/2}}$$

and $f/g \to 1$ for $a_1 \to \infty$.

A similar calculation can be done for the pure shear case.

The above result can be applied to a flat ellipsoidal crack ($a_3 \to 0$) since (9) does not contain a_3. For the slit-like crack ($a_1 \to \infty$), G agrees with that given by Barnett and Asaro [3].

STRESS CONCENTRATION FACTORS

The stress concentration factor is defined by

$$\kappa = \left(\sigma^0_{ij} + \sigma_{ij}(\mathrm{out})\right)/\sigma^0_{ij} \tag{11}$$

where $\sigma^0_{ij} + \sigma_{ij}(\mathrm{out})$ is defined immediately outside the crack. $\sigma_{ij}(\mathrm{out})$ can be determined by σ_{ij} in Ω (denoted by $\sigma_{ij}(\mathrm{in})$). The stress jump on Ω, $[\sigma_{ij}] \equiv \sigma_{ij}(\mathrm{out}) - \sigma_{ij}(\mathrm{in})$, can be written as (see [4] - [7]),

$$\left[\sigma_{ij}\right] = C^0_{ijk\ell}\left\{-C^0_{pqmn}\varepsilon^*_{mn}G_{kpq\ell}(\underset{\sim}{n}) + \varepsilon^*_{k\ell}\right\} \tag{12}$$

where n is the outward normal vector of the boundary of Ω. Since $\sigma^0_{ij} + \tilde{\sigma}_{ij}(\mathrm{in}) = 0$, (11) is written as

$$\kappa = \left[\sigma_{ij}\right]/\sigma^0_{ij} \tag{13}$$

where ε^*_{ij} are given by (5) and (6) for the two types of applied stress and are inversely proportional to a_3. κ at $\beta = \pi/2$ (see Figure 1) is shown in Figures 2 and 3 with respect to a_1/a_2 for various crystals. a_3 is expressed in terms of ρ_n from geometry,

$$a_3 = (\rho_n a_1 a_2)^{1/2}(a_2^2\cos^2\beta + a_1^2\sin^2\beta)^{-1/4} \tag{14}$$

where ρ_n is the root radius of the curve which is an intersection of Ω and the plane containing n and the line parallel to the x_3 axis (see Figure 1). The values of $\kappa/(a_2/\rho_{\tilde{n}})^{1/2}$ converge to constant values which agree with the values expected from Lekhnitiskii [8] for simple tension in the plane strain case.

STRESS INTENSITY FACTORS

The stress intensity factor is defined when $a_3 \to 0$ by

$$K = \lim_{\rho\to 0}\sqrt{2\rho}\left(\sigma^0_{ij} + \sigma_{ij}\right) \tag{15}$$

where ρ is the distance from the boundary of the crack.

When a_1 is fixed and a_2 is changed by δa_2 (small variation), the work done by the tension on the new crack surface δA is

$$2(K/\sqrt{2\rho})\varepsilon^*_{33}h\delta A = 2K\varepsilon^*_{33}a_3(\bar{x}_1^2/a_1^4 + \bar{x}_2^2/a_2^4)^{1/4}\pi a_1\delta a_2 \tag{16}$$

where $\bar{x}_1^2/a_1^2 + \bar{x}_2^2/a_2^2 = 1$ and $(1 - x_1^2/a_1^2 - x_2^2/a_2^2)^{1/2}$ is approximated as $\sqrt{2\rho}(\bar{x}_1^2/a_1^4 + \bar{x}_2^2/a_2^4)^{1/4}$. Equation (16) must be equal to $G\delta a_2$ which can be written from (5) as

$$G\delta a_2 = 2\pi a_1 \sigma_{33}^0 a_3 \varepsilon_{33}^* (f/g)\delta a_2. \tag{17}$$

Comparing (16) and (17), we have

$$K = \sigma_{33}^0 (\bar{x}_1^2/a_1^4 + \bar{x}_2^2/a_2^4)^{-1/4}(f/g) \tag{18}$$

For the slit-like crack ($a_1 \to \infty$), $f/g = 1$ and therefore K is independent of the elastic constants. The value of K agrees with Barnett and Asaro [3] for the slit-like crack. Similar discussion can be done for the pure shear case.

CONCLUSION

We have shown that the equivalent inclusion method can provide the crack opening displacements, crack extension forces, the stress concentration factor when $a_3 \neq 0$, and the stress intensity factor when $a_3 = 0$. It is also found that some difficulties are involved in deriving the stress intensity factors from the stress concentration factors through a limiting process. The two concepts of stress intensity and concentration factors appear to belong to separate categories.

In this paper we considered uniform applied stress fields at infinity. However, a similar calculation can be done for a linearly changing applied stress at infinity. Similarly the present method can be applied to any orientation of crystals and any anisotropic material. Comparison with other results about the stress intensity factors obtained by Kassir and Sih [9] and Willis [10] will be possible only numerically and will be reported in this conference.

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APPENDIX

For isotropic materials

$$C_{33mn}^0 C_{pq33}^0 \Pi_{mpnq} = \frac{1}{a_2}\frac{4\pi\mu}{1-\nu}E(k)$$

$$C_{31mn}^0 C_{pq31}^0 \Pi_{mpnq} = \frac{1}{a_2} 4\pi\mu\left\{\frac{1}{1-\nu}\frac{k'^2}{k^2}(F(k)-E(k)) + \frac{k'^2}{k^2}\left(\frac{E(k)}{k'^2} - F(k)\right)\right\}$$

where μ, ν are the shear modulus and Poisson's ratio, respectively, and $k^2 = 1 - a_2^2/a_1^2$, $a_1 > a_2$, $k'^2 = 1 - k^2$. $F(k)$, $E(k)$ are the complete elliptic integrals of the first and second kinds respectively.

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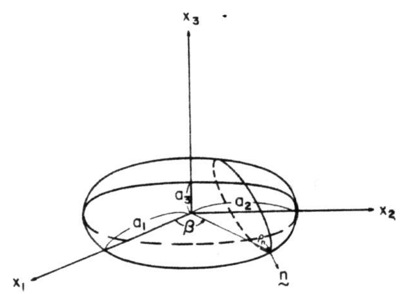

Figure 1 Configuration of a Flat Ellipsoid or a Crack

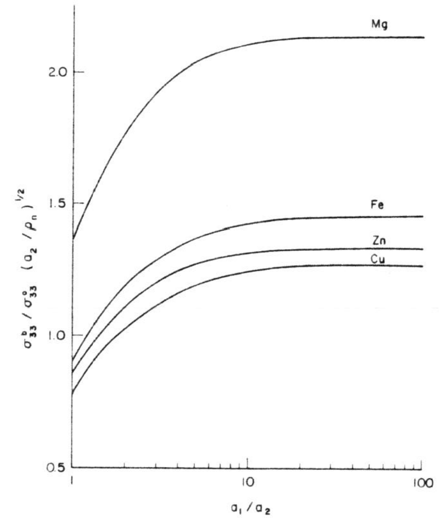

Figure 2 Tensile Stress on the Boundary of an Elliptical Crack at $\beta = 90°$ for Various Ratio of a_1/a_2

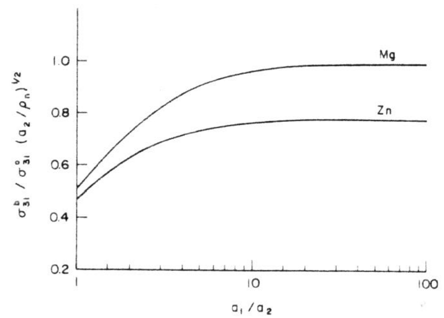

Figure 3 Shear Stress on the Boundary of an Elliptical Crack at $\beta = 90°$ for Various Ratio of a_1/a_2