PROBABILITY OF FATIGUE FAILURE BASED ON RESIDUAL STRENGTH

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INTRODUCTION

The residual strength of a structure, given by the maximum load it can support before failure, has served as a useful parameter in many fatigue reliability studies. It is usually assumed that the initial static strength of a structure reduces with the number of cycles applied and failure occurs when it equals the maximum stress in the fatigue loading. It should, however, be recognized that, although the residual strength changes continuously with the number of cycles applied, it represents weakening of the material due to microscopic cracks in the crack initiation stage, while it stands for the stress at which the growth of a predominant crack becomes unstable in the crack propagation stage. It can therefore be expected that, in these two stages of the fatigue process, the probability distribution of the residual strength will have different properties.

In the following, it will be shown that, if the initial strength of a material follows the Weibull probability distribution, then the residual strength, after a given number of cycles, also follows this distribution. Furthermore, in each stage of the fatigue process, the shape parameter of the distribution remains unchanged, while the location and scale parameters decrease with the number of cycles applied. Test results for specimens of a steel alloy clearly indicate that the one-component distribution of the initial strength develops into a two-component distribution of the residual strength after fatigue cycling. The component with high residual strength corresponds to the crack initiation stage and that with low residual strength to the crack propagation stage.

RESIDUAL STRENGTH

Crack Initiation Stage

In the crack initiation stage, the weakening of strength occurs due to microcracks generated at strain incompatibility centres spread throughout the cross-section of a specimen. This weakening can be characterized by a damage parameter \( D \) with its value \( D=0 \) in the virgin state and \( D=1 \) at failure. Expressing \( D \) in terms of the residual strength \( R \), we have

\[
D = \frac{(R_0 - R)}{(R_0 - S)} \tag{1}
\]

where \( R_0 \) is the initial strength and \( S \) is the maximum stress applied.

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Assuming further that the damage per cycle is stress dependent and is given by

$$\frac{d\bar{b}}{dn} = n (S/(1-D))^m$$

(2)

we have, from equations (1) and (2)

$$\frac{d\bar{R}}{dn} = -n S (R_S)^{1+m}/(R-S)^n$$

(3)

Integrating equation (3) we get a relation between the initial and the instantaneous residual strength:

$$R_0 = S + (R-S)/(1-n(1+m)S^n)^{1/(1+m)}$$

(4)

If the probability distribution of the initial strength is given by

$$P(R_0 \leq r) = F(r)$$

(5)

then from equations (4) and (5) we observe that, for a given \(N\), the probability of failure \(P_F\) will be given by

$$P_F = P(R \leq S) = P\left[R_0 \leq S + (R-S)/(1-n(1+m)S^n)^{1/(1+m)}\right]$$

$$= P\left[S + (R-S)/(1-n(1+m)S^n)^{1/(1+m)}\right]$$

(6)

Assuming the Weibull distribution for the initial strength, we have

$$F(r) = 1 - \exp\left(-\left(\frac{r-a}{b}\right)^c\right)$$

(7)

where \(a\), \(b\) and \(c\) are the location, scale and shape parameters, respectively.

From equation (6) we have

$$P(R \leq S) = 1 - \exp\left(-\left(\frac{a-a'}{b'}\right)^c\right)$$

(8)

where

$$a' = S + (a-S)/(1-n(1+m)S^n)^{1/(1+m)}$$

(9)

and

$$b' = b(1-n(1+m)S^n)^{1/(1+m)}$$

(10)

From equation (8) we see that the shape parameter remains unchanged, while the location and scale parameters decrease with the number of cycles applied as shown by equations (9) and (10).

**Crack Propagation Stage**

In the crack propagation stage, the residual strength is given by

$$R(C) = K_C \alpha$$

(11)

where \(C\) is the crack length and \(\alpha\) is a constant.

For plain strain crack growth, the crack growth rate has been found to be proportional to the crack length for most of the crack propagation stage [1]. We therefore have

$$\frac{dC}{dn} = bC$$

(12)

where \(b\) is a constant given by

$$b = \gamma S^D$$

(13)

Following the procedure outlined for the crack initiation stage, we get

$$P(R \leq S) = 1 - \exp\left(-\left(\frac{a''}{b''}\right)^c\right)$$

(14)

where

$$a'' = a \exp\left(-\frac{1}{2} \gamma S^D\right)$$

(15)

and

$$b'' = b \exp\left(-\frac{1}{2} \gamma S^D\right)$$

(16)

**TEST PROGRAM**

Specimens were prepared from 22mm dia. bars of a Cr-Mo-V steel, which were machined to give a smooth reduction to 10mm x 15mm cross-section. On one 15mm side a small surface flaw was introduced by electro-discharge machining. The flaw was 0.15mm on the surface and 0.05mm in depth. The processes of machining and introducing flaws were carried out in randomized order.

The test program consisted of 6 test series. In each series, a pre-selected number of specimens were drawn randomly from the lot containing all specimens. The specimens in Series No. 1 were pulled in tension to estimate the initial strength. The specimens in Series Nos. 2, 3, 4 and 5 were subjected to pre-specified number of cycles at the maximum stress of 380 MPa and the minimum stress of -235 MPa. The unfailed specimens were then pulled in tension to estimate the residual strength. The specimens in Series No. 6 were subjected to fatigue cycling until failure.

Table 1 shows the residual strength data for the unfailed specimens and the fatigue life in cycles for the failed specimens.

**STATISTICAL ANALYSIS OF TEST DATA**

**Methods of Analysis**

The test data is analyzed by means of two plots: the \((x_1, y_1)\) plot and the \((S_k, k^{-1/2})\) plot.
The \((x_i, Ez_i)\) Plot

The standardized variable \(z\) is defined by

\[
z = \frac{(x-a)}{b}
\]  
(17)

where \(x\) is the random variable in the Weibull distribution.

The expected value of the order statistic \(z_i\), denoted by \(Ez_i\), is given by

\[
Ez_i = (\frac{1}{c})^{1+1} \cdot 1 \cdot (-1)^{\frac{1}{c}} \cdot (1+1/c)
\]  
(18)

As can be seen in this equation, \(Ez_i\) is, for a given size of the sample \(M\), a function of the shape parameter \(c\) only.

If now the observations are denoted by \(x_i\), the parameters \(a\) and \(b\) can be estimated by use of the relation

\[
x_i = a + b \cdot Ez_i
\]  
(19)

Thus, if \(x_i\) is plotted against \(Ez_i\), the data points \((x_i, Ez_i)\) will, with due regard to "the sampling scatter", fall on a straight line. The parameter \(c\) is estimated by fitting a regression line to the data points by means of the least-square principle for different values of \(c\) and accepting that value of \(c\) which provides the least sum of the squared deviations of the fitted line.

The \((S_k, k^{-1/c})\) Plot

By grouping independent observations randomly into a number of subsamples, a new statistic, denoted by \(S_k\) is obtained. The relation of this statistic to the order statistic \(x_i\), derived in a work to be published, is given by

\[
S_k = \frac{M^{-1-k}}{k^{-1}} \cdot x_i
\]  
(20)

where

\[
r_{ki} = \frac{M^{-1-k}}{k^{-1}} / \frac{M^{-1-k}}{k^{-1}}
\]  
(21)

If the observed values of \(S_k\), i.e. the linear function of the observations \(x_i\) given by equation (20), is equated to its expected value, we arrive at the relation

\[
S_k = a + b(1/c)(k^{-1/c})
\]  
(22)

which provides another graphical method for estimating the distribution parameters.
Table 1  Test Data. Residual Strength $x_i$ in MPa and Fatigue Life $N_i$ in number of cycles

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Figure 1  The $(x_i,E_{ij})$ Plot. Circles: Series Nos. 1-2, triangles: Series No. 3, inverted triangles: Series No. 4, squares: Series No. 5.