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NECKING OF ELASTIC-PLASTIC CYLINDERS UNDER UNIAXIAL TENSION

R. N. Dubey and A. H. Elkholy*

INTRODUCTION

In tensile deformation of metals which exhibit a pronounced acceleration in tensile creep, readily detectable cavities occur in the necked zone. It is not clear at what stage of the necking process formation of cavities occurs, but the process itself is terminated by necking rupture. The initiation of necking would thus appear to be the key to failure by rupture and therefore it is necessary to understand the mechanism for the onset of necking. An understanding of this process is also needed in the development of models for the internal linkage of cavities.

Recent papers by Ariaratnam and Dubey [1], Dubey and Ariaratnam [2], Cheng, Ariaratnam and Dubey [3], Miles [4], and Hutchinson and Miles [5] throw some light on the mechanism of necking of materials whose constitutive properties are governed by Prandtl-Reuss equations. There are other problems of practical importance, plates under biaxial tension for example, which still need to be tackled satisfactorily.

Necking solutions for rectangular plates under uniaxial and biaxial tension have been obtained by Dubey and Ariaratnam [6] and Miles [7]. The values of the necking stress obtained in [6] by neglecting the incremental elastic strain energy is found to be attainable by metals. Moreover, these values are lower than the corresponding values at the maximum load. An exact solution for a square plate under biaxial tension obtained in [2] yields a value for necking stress which is of the order of elastic moduli and hence of no practical consequence. Miles [7] modified some of the boundary conditions and obtained necking stress for square plates under biaxial tension which is attainable for metals. He also points out (see also [5]) that the initiation of necking can occur only after the maximum load has been reached. This observation would appear to make any study of necking useless from a practical point of view and hence of academic interest only.

Recently, Dubey [8] has pointed out some inconsistencies in the Prandtl-Reuss equations. Subsequently, the constitutive properties were modified in [9] to remove these inconsistencies. The modified constitutive relations have been used to develop a criterion for uniqueness in [10]. The uniqueness criterion has been used by Elkholy [11] to obtain bifurcation stress for square plates under biaxial tension. It turns out that the first necking stress, we shall call it the primary necking stress, is lower than the stress σ_{max} at the maximum load. In fact, in all cases examined, σ_{max} has been found to be an upper bound for the primary necking stress. Other values of necking stress higher than the primary value have been observed, some of these occur beyond σ_{max}. These stresses are of academic interest only.

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* Department of Mechanical Engineering, University of Waterloo, Waterloo, Ontario, Canada, N2L 3G1

In this work, the uniqueness criterion of [10] is used to obtain necking stress for an incompressible and isotropic elastic-plastic cylinder under uniaxial tension. An exact solution shows that the primary necking stress is lower than the stress at the maximum load.

CONSTITUTIVE PROPERTIES

Consider an incompressible elastic-plastic solid whose material behaviour is isotropic in both elastic and plastic deformations. Suppose that the solid deforms from its initial stress-free configuration $\mathcal{B}_0$ to a current configuration $\mathcal{B}$ under a prescribed external loading. Let σ_{ij} be the components of the Cauchy stress and e^e_{ij} be the associated elastic strain components on fixed Cartesian coordinates x_i.

In view of the assumed incompressibility and isotropy, the deviatoric stress, $\sigma'_{ij} = \sigma_{ij} - (1/3)\,\sigma_{kk}\,\delta_{ij}$, where δ_{ij} is Kronecker delta and e^e_{ij} are coaxial and proportional. Simply stated

$$\sigma'_{ij} = 2\mu\, e^e_{ij} \,, \tag{1}$$

where 2μ is the elastic shear modulus.

If the body undergoes incremental deformation from $\mathcal{B}$ to a final configuration $\mathcal{B}'$ causing, in the process, infinitesimal rotations of the principal axes of the stress, then the principal axes of the elastic strain must also undergo the same rotations. That is

$$\sigma'_{ij} + d\sigma'_{ij} = 2\mu\,(e^e_{ij} + de^e_{ij}) \tag{2}$$

in $\mathcal{B}'$.

In other words, the elastic strain in any state must be coaxial with and proportional to the deviatoric part of the stress causing this strain.

Let us assume that the solid obeys von Mises yield criterion

$$\sigma = \left(\frac{3}{2}\,\sigma'_{ij}\,\sigma'_{ij}\right)^{1/2} \tag{3}$$

where σ is the yield stress in simple tension. For the plastic deformation to occur during the motion $\mathcal{B} \to \mathcal{B}'$, the current stress must satisfy (3) and, in addition, the stress increment must be such that

$$d\sigma = (3/2\sigma)\,\sigma'_{ij}\,d\sigma'_{ij} > 0 \,. \tag{4}$$

In short, the plastic flow is caused by stress $\sigma'_{ij} + d\sigma'_{ij}$ which must be such as to satisfy both (3) and (4). For isotropic plastic deformation, the plastic strain-increment de^p_{ij} must be coaxial with and proportional to

$\sigma'_{ij} + d\sigma'_{ij}$ causing the plastic flow. That is,

$$de^p_{ij} = \left(\sigma'_{ij} + d\sigma'_{ij}\right) d\lambda \tag{5}$$

where $d\lambda$ is a constant which depends on the history of deformation and vanishes with $d\sigma$. (See Appendix).

For infinitesimal incremental deformation, we can assume

$$de_{ij} = de^e_{ij} + de^p_{ij} \,. \tag{6}$$

Combining (2) and (5) with the help of (6), we obtain;

$$\sigma'_{ij} + d\sigma'_{ij} = 2\bar{\mu}\left(e^e_{ij} + de_{ij}\right) \tag{7}$$

where,

$$2\bar{\mu} = 2\mu/(1 + 2\mu d\lambda) \tag{8}$$

is the effective shear modulus.

Finally, the stress in $\mathcal{B}'$ can be expressed as

$$\sigma_{ij} + d\sigma_{ij} = (1/3)(\sigma_{\kappa\kappa} + d\sigma_{\kappa\kappa})\sigma_{ij} + 2\bar{\mu}(e^e_{ij} + de_{ij}) \,. \tag{9}$$

STATEMENT OF THE PROBLEM AND RESULTS

Consider a cylindrical bar of radius a and length L subjected to uniaxial tension along the cylindrical axis. Let us assume that the current stress distribution is homogeneous throughout the volume V of the body. During the finite deformation $\mathcal{B}_0$ to $\mathcal{B}$, the cylindrical axis is also the principal axes for both σ_{ij} and e^e_{ij}. During the homogeneous incremental deformation from $\mathcal{B}$ to $\mathcal{B}'$, the principal axes of $\sigma_{ij} + d\sigma_{ij}$, $e^e_{ij} + de^e_{ij}$ and de^p_{ij} coincide with the cylindrical axis. We now look at the possibility of another, possibly noncoaxial, mode of deformation under the same prescribed boundary conditions. The difference between the two modes of deformation must satisfy the homogeneous equations of equilibrium and boundary conditions. (These conditions are similar to equation (7) and (4) in [3]). The solution for the difference fields is obtained in a manner similar to the one used in [3]. Here, we present the condition for the existence of a non-trivial solution for the difference field as the condition for bifurcation of equilibrium as

$$F_1 = F_2 \,. \tag{10}$$

In (10),

$$F_1 = \alpha a \; I_o(\alpha a)/I_1(\alpha a) \tag{11}$$

$$F_2 = \frac{\sigma}{2\bar{\mu}} + (1 - \frac{\sigma}{2\bar{\mu}})^2 \; \alpha\rho a \; I_o(\alpha\rho a)/I_1(\alpha\rho a) \quad \text{if } 2\bar{\mu} > \sigma \tag{12}$$

$$F_2 = \frac{\sigma}{2\bar{\mu}} + (\frac{\sigma}{2\bar{\mu}} - 1)^2 \; \alpha\rho a \; J_o(\alpha\rho a)/J_1(\alpha\rho a) \quad \text{if } 2\bar{\mu} < \sigma$$

$$\rho = \sqrt{\left|\frac{2\bar{\mu}+\sigma}{2\bar{\mu}-\sigma}\right|} \tag{13}$$

where $\alpha a = 2\pi a/\ell$. The half-wave length ℓ in F_1 and F_2 need not be the length of the specimen L. $J_p(x)$ and $I_p(x)$ are the Bessel function and the modified Bessel function of order p, respectively.

The necking stress is obtained from (10) for a solid stress whose stress-strain relationship is given by

$$e = \frac{\sigma}{E} + A\sigma^{\beta} \; . \tag{14}$$

E is the Young's modulus = 73.538 x 10^9 pa, A = 1.167 x 10^{-76} and β = 8.6127 are two material constants. The ratio of the increase in traction dT required to sustain the incremental coaxial mode to the traction T in B was taken equal to 0.001, 0.005, 0.01 and 0.05 and yielded necking stresses of 509,811 x 10^3 pa, 509,736 x 10^3 pa, 509,632 x 10^3 pa and 508,584 x 10^3 pa, respectively. These values of necking stresses are first to be encountered for increasing value of the stress σ and hence are the primary necking stresses for each dT/T. Each of the primary necking stresses are lower than the stress σ_{max} = 509,832 x 10^3 pa at the maximum load. Necking stresses higher than the above values have been observed but are not quoted here as they are not likely to be of any practical importance.

APPENDIX

Calculation of the Parameter $d\lambda$

Let

$$d\lambda = C d\sigma \; , \tag{A1}$$

where the constant C depends upon the prior history of deformation and on the yield criterion.

Define

$$de^p \equiv (\frac{2}{3} \, de^p_{ij} \, de^p_{ij})^{1/2} \tag{A2}$$

Using (A2), (A1), (3), (4) and (5), we obtain

$$(1/C) = \frac{2}{3} \, \sigma \, \frac{\partial\sigma}{\partial e^p} \; . \tag{A3}$$

In deriving (A3) we assumed $d\sigma << \sigma$.

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