A Refined Theory of Spherical Shells with Cracks

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It is now well known in the literature on fracture mechanics that the approximate Kirchoff boundary conditions [1-3] used in the classical plate theory are not adequate for determining the stress and displacement fields in the neighborhood of the crack tip. This is mainly because of the omission of strain energy of the transverse shear stresses in the derivation of classical shell theory which contracts five of the natural boundary conditions into four. In this work an improved theory of shallow spherical shells which includes the effect of transverse shear deformation is derived. The resulting tenth order system of equations are uncoupled and all five boundary conditions along an edge of the shell can be satisfied [4].

The method of integral transforms is used to formulate and solve the symmetric problem for a spherical shell containing a finite meridional crack. The stress field in the neighborhood of the crack tip is obtained:

$$\tau_{xx} = [k_1(z)/\sqrt{2r}] \cos(\theta/2)[1 - \sin(\theta/2)\sin(3\theta/2)] + O(1)$$

$$\tau_{yy} = [k_1(z)/\sqrt{2r}] \cos(\theta/2)[1 + \sin(\theta/2)\sin(3\theta/2)] + O(1)$$

$$\tau_{xy} = [k_1(z)/\sqrt{2r}] \cos(\theta/2)\sin(\theta/2)\cos(3\theta/2) + O(1)$$

$\tau_{xz} = \tau_{yz} = 0(1)$ (1)

where τ_{xx}, τ_{yy}, τ_{xy} are the in-plane stresses and τ_{xz}, τ_{yz} are the transverse shear stresses in the shell. In contrast to the classical results, the angular variation of the membrane and bending stresses coincide and hence a combined stress-intensity factor can be defined. The $1/\sqrt{r}$ singularity remains unchanged. As usual, r and θ represent the polar coordinates measured from the right-hand side crack tip. Furthermore, it should be mentioned that in the shell problem, both the extensional and bending effects are always coupled regardless of the nature of the loading. The numerical results for two specific cases will now be discussed:

1. Extensional Load: $N(x) = N_0$ and $M(x) = 0$

Let the crack surfaces be opened out by uniform extensional loads of constant magnitude N_0. The loading situation is illustrated in Figure 1. Of interest are the stress intensity factors at the top and bottom surface layers $z = \pm h/2$ of the shell as given by

$$k_1^{(e)}(\pm h/2) = N_0\sqrt{a}[\psi_1^{(e)}(1)\pm\psi_2^{(e)}(1)]/[1\pm(h/2R)]h \qquad (2)$$

for the case of extensional load which is identified by superscript (e). The functions $\psi_1^{(e)}$ and $\psi_2^{(e)}$ can be computed numerically from a system of coupled Fredholm integral equations of the second kind. In equation (2), a is the half crack length, h the shell thickness and R the radius of curvature of the shell. The numerical values of $k_1^{(e)}$ at the bottom layer of the shell ($z = -h/2$) are shown graphically in Figure 1 as a function of h/2a for a Poisson's ratio of $\nu = 1/3$ and $h/R = 0.1$. Note that $k_1^{(e)}$ decreases with h/2a

and the refined theory predicts larger values of $k_1^{(e)}$. The difference becomes more pronounced for longer cracks, i.e., as $h/2a \to 0$. It should be cautioned that the results corresponding to short cracks are not valid since they violate the assumption of the theory of thin shells.

2. Bending Load: $M(x) = M_0$ and $N(x) = 0$

When the loading on the crack surfaces are of the bending type, say equal and opposite uniform moments with magnitude M_0, the results are quite different. In this case, the stress intensity factor for $z = \pm h/2$ is given by

$$k_1^{(b)}(\pm h/2) = \pm 6M_0\sqrt{a}[\psi_2^{(b)}(1)\pm\psi_1^{(b)}(1)]/[1\pm h/2R)]h \qquad (3)$$

Under bending, one side of the crack tends to close as it is subjected to compression. The maximum value of $k_1^{(b)}$ will always occur on the tension side. With the tension side at $z = h/2$, Figure 2 gives a plot of $h^2k_1^{(b)}(h/2)/6M_0\sqrt{a}$ versus h/2a for $\nu = 1/3$ and $h/R = 0.1$, 0.2 and 0.3. A dramatic departure of the present results from those obtained by the classical theory is shown where the trend of the variations of $k_1^{(b)}$ with the h/2a is, in fact, opposite. The difference becomes more and more significant as the ratio h/2a decreases. The results based on the classical theory is known to be approximate since the boundary conditions on the crack surfaces are satisfied only in the Kirchhoff sense. A more detailed discussion on the approximate nature of the classical shell theory has been given by Sih and Dobreff [2] and will not be repeated here.

Presented in this paper is a refined theory for shallow spherical shells which incorporates the strain energy stored in the shell due to shear deformation. This results in a system of equations that are compatible with five distinct physical boundary conditions to be satisfied on a free edge as compared to four conditions in the classical theory. In dealing with the crack problems, which are concerned with the effect of free surfaces in the material, the additional boundary condition leads to more realistic results since they possess the character of the three-dimensional crack-tip stress field [5]. The information gained here is pertinent to the strength analysis of imperfect shell structures. Within the framework of the classical fracture mechanics concept, it is possible to estimate the fracture strength of a shell by knowing the toughness of the material which is indicative of the energy required to cause the propagation of a flaw of pre-determined size.

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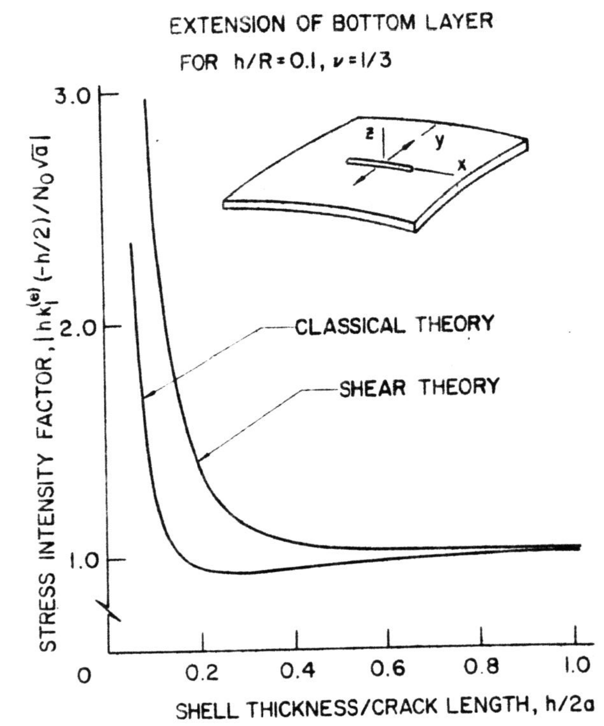

Figure 1 - Extensional Load on Crack

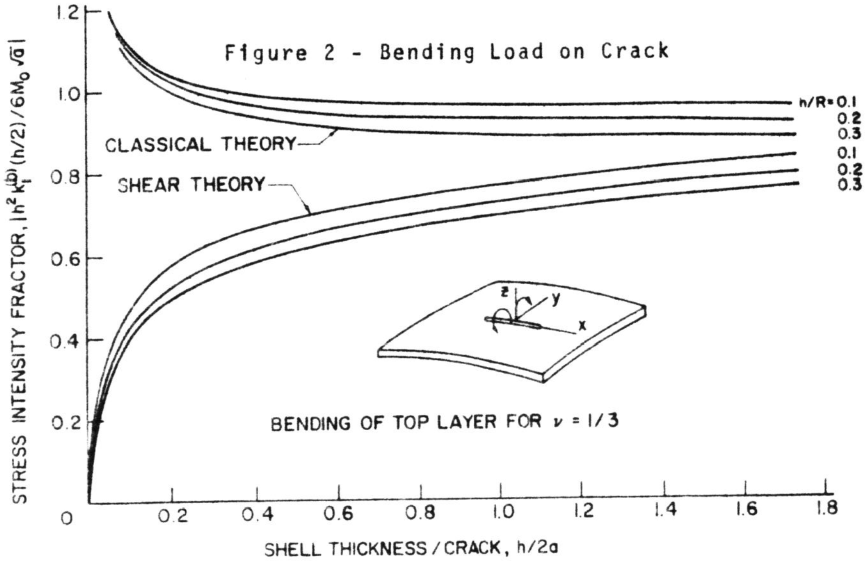

Figure 2 - Bending Load on Crack