A Study of Crack Paths of Two-dimensional Multi-fibre Microcomposites

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Abstract- *A new model is presented for analyzing the stress concentration of unidirectionally reinforced fiber composite. The present model is based on variational theory. The stress components are derived in terms of a perturbation function. A parametric study of stress concentration factor (SCF) of the unidirectionally reinforced fiber composite is carried out. In order to validate the present model, the numerical results from the present work compare with those obtained from the shear lag model. There is a good agreement between the results obtained by the two models.*

Keywords: Stress concentrations; Composites; Fiber break; Variational solution

1 Introduction

Having better stiffness, strength and impact resistance, reinforced fiber composite material has been widely using in the aerospace, automobile, marine industries and civil structures. The longitudinal tensile failure of unidirectionally reinforced fiber composite material is complex. Initially all fibers are intact and able to carry load. With the increase of tensile load, a fiber first breaks at a weakest point. This break is at random position, and is due to flaws. Fiber break creates stress concentrations in the adjacent fibers. The stress concentrations may cause the progress failure of fibers

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and eventually the failure of the whole composite. The stress concentrations are affected by many factors such as the spacing of the fibers, the elastic properties of fiber and matrix, the characteristic of fiber/matrix interface, and so on. It is important to investigate the effects of these factors on the stress concentrations around fibers break in unidirectionally reinforced fiber composite.

Predicting the fracture behavior of unidirectionally reinforced fiber composite has been trackled in several ways including full elasticity solutions for idealized cases, various models that build on classical shear lag analysis and finite element methods. The shear lag model was developed by Cox,[1] Hedgeprth,[2] and Greszczuk,[3]. Fukuta and Kawata,[4] incorporated the material properties of fiber and matrix into the calculation of the SCF. Wagner and Eitan,[5] defined an effective radius around the fiber break to calculate the SCF. Grubb et al,[6] modified Wagner and Eitan' model. They uesed the experimentally determined fiber/fiber interaction radius to calculate the SCF. In the past years, 2D and 3D finite element computations have been done in sveral studies,[7,8] to analyze the stress concentrations caused by a fiber break. These calculations allow us to correctly evaluate the stress concentrations.

However, alomost all the analysis on the stress concentration around fiber break in unidirectionally reinforced fiber composite built on the shear lag theory. In the present work, the stress concentration is studied by using the variational theory. The influences of the spacing of the fibers, the elastic properties of fiber and matrix to the stress concentration of unidirectionally reinforced fiber composite caused by fiber breaks are investigated in the paper.

2 Theoretical modelling

The model of a 2D unidirectionally reinforced fiber composite studied in this paper is shown in Fig.1. Φ and s denote the fiber radius and the inter-fiber spacing, respectively. The length of the lamina is denoted by L. The origin for the rectangular Cartesian coordinates (x, y) is selected to the mid-way of the lamina. From the symmetry, it is only necessary to consider a quarter $(0 \le x \le L/2, y \ge 0)$ of the lamina as illustrated in Fig.2, which can be divided into four regions. Region Ⅰ and Region Ⅲ are the fiber areas. Region Ⅱ and Region Ⅳ are the matrix areas.

In the present work, we assume that the normal stress component in the longitudinal direction of the present model is of the form:

$$\sigma_{xx}^{(\mathrm{I})} = \sigma_{xx_0}^{(\mathrm{I})} + \sigma_{xx_0}^{(\mathrm{I})}\psi(x) \qquad 0 \le x \le 0.5L, \quad 0 \le y \le 0.5\Phi \tag{1}$$

$$\sigma_{xx}^{(\mathrm{II})} = \sigma_{xx_0}^{(\mathrm{II})} + \sigma_{xx_0}^{(\mathrm{II})}\varphi(x) \qquad 0 \le x \le 0.5L, \quad 0.5\Phi \le y \le 0.5\Phi + s \tag{2}$$

$$\sigma_{xx}^{(\mathrm{III})} = \sigma_{xx_0}^{(\mathrm{III})} + \sigma_{xx_0}^{(\mathrm{III})}\varphi(x) \qquad 0 \le x \le 0.5L, \quad 0.5\Phi + s \le y \le 1.5\Phi + s \tag{3}$$

$$\sigma_{xx}^{(\mathrm{IV})} = \sigma_{xx_0}^{(\mathrm{IV})} + \sigma_{xx_0}^{(\mathrm{IV})}\varphi(x) \qquad 0 \le x \le 0.5L, \quad 1.5\Phi + s \le y \le 1.5\Phi + 2s \tag{4}$$

where $\varphi\ (x)$ and $\psi\ (x)$ are unknown functions which needs to be determined and

$$\sigma_{xx_0}^{(\mathrm{I})} = \sigma_{xx_0}^{(\mathrm{III})} = \frac{N_x E_f}{1.5\Phi E_f + 2sE_m} \tag{5}$$

$$\sigma_{xx_0}^{(\mathrm{II})} = \sigma_{xx_0}^{(\mathrm{IV})} = \frac{N_x E_m}{1.5\Phi E_f + 2sE_m} \tag{6}$$

are the longitudinal normal stress components of the fiber and matrix in the lamina under the tensile load N_x. Here, E_f and E_m are Young's moduli of the fiber and matrix.

Along the line of Hashin, McCartney and Gao's analysis [9-11], the relation between

$\varphi\ (x)$ and $\psi\ (x)$ is determined using the global equilibrium condition

$$N_x = \int_0^{\frac{\Phi}{2}} \sigma_{xx}^{(\mathrm{I})} dy + \int_{\frac{\Phi}{2}}^{\frac{\Phi}{2}+s} \sigma_{xx}^{(\mathrm{II})} dy + \int_{\frac{\Phi}{2}+s}^{\frac{3\Phi}{2}+s} \sigma_{xx}^{(\mathrm{III})} dy + \int_{\frac{3\Phi}{2}+s}^{\frac{3\Phi}{2}+2s} \sigma_{xx}^{(\mathrm{IV})} dy \tag{7}$$

Substituting Eqs.(1)-(6) into Eq.(7) yields

$$\psi(x) = -\frac{\Phi E_f}{4sE_m + 2\Phi E_f}\varphi(x) \tag{8}$$

For the plane problem, the equilibrium equations of the lamina are

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} = 0 \tag{9a}$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \tag{9b}$$

Substituting Eqs.(1)-(4) in Eq.(9a), σ_{xy} is obtained by integration

$$\sigma_{xy}^{(\mathrm{I})} = -\sigma_{xx_0}^{(\mathrm{I})}\psi'(x)y + f_1(x) \tag{10}$$

$$\sigma_{xy}^{(\mathrm{II})} = -\sigma_{xx_0}^{(\mathrm{II})}\varphi'(x)y + f_2(x) \tag{11}$$

$$\sigma_{xy}^{(\mathrm{III})} = -\sigma_{xx_0}^{(\mathrm{III})}\varphi'(x)y + f_3(x) \tag{12}$$

$$\sigma_{xy}^{(\mathrm{IV})} = -\sigma_{xx_0}^{(\mathrm{IV})}\varphi'(x)y + f_4(x) \tag{13}$$

where $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$ are unknown functions.

Inserting Eqs.(10)-(13) into Eq.(9b) yields

$$\sigma_{yy}^{(\mathrm{I})} = \frac{1}{2}\sigma_{xx_0}^{(\mathrm{I})}\psi''(x)y^2 - f_1'(x)y + g_1(x) \tag{14}$$

$$\sigma_{yy}^{(\mathrm{II})} = \frac{1}{2}\sigma_{xx_0}^{(\mathrm{II})}\varphi''(x)y^2 - f_2'(x)y + g_2(x) \tag{15}$$

$$\sigma_{yy}^{(\mathrm{III})} = \frac{1}{2}\sigma_{xx_0}^{(\mathrm{III})}\varphi''(x)y^2 - f_3'(x)y + g_3(x) \tag{16}$$

$$\sigma_{yy}^{(\mathrm{IV})} = \frac{1}{2}\sigma_{xx_0}^{(\mathrm{IV})}\varphi''(x)y^2 - f_4'(x)y + g_4(x) \tag{17}$$

where $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$ are unknown functions.

The boundary conditions for the quarter of the lamina shown in Fig.3 include the traction-free conditions on the top surface:

$$\sigma_{yy}^{(IV)}(x,h) = \sigma_{xy}^{(IV)}(x,h) = 0 \tag{18a,b}$$

where $h = 1.5\Phi + 2s$

The symmetry condition in the middle plane:

$$\sigma_{xy}^{(I)}(x,0) = 0 \tag{19}$$

The traction-free conditions on the crack surface:

$$\sigma_{xx}^{(I)}\left(\frac{L}{2}, y\right) = \sigma_{xy}^{(I)}\left(\frac{L}{2}, y\right) = 0 \tag{20a,b}$$

The traction continuity conditions on the interface $y = 0.5\Phi$:

$$\sigma_{yy}^{(I)}(x,0.5\Phi) = \sigma_{yy}^{(II)}, \quad \sigma_{xy}^{(I)} = \sigma_{xy}^{(II)} \tag{21a,b}$$

The traction continuity conditions on the interface $y = 0.5\Phi + s$:

$$\sigma_{yy}^{(II)}(x,0.5\Phi + s) = \sigma_{yy}^{(III)}(x,0.5\Phi + s),\ \sigma_{xy}^{(II)}(x,0.5\Phi + s) = \sigma_{xy}^{(III)}(x,0.5\Phi + s) \tag{22a,b}$$

The traction continuity conditions on the interface $y = 1.5\Phi + s$:

$$\sigma_{yy}^{(III)}(x,1.5\Phi + s) = \sigma_{yy}^{(IV)}(x,1.5\Phi + s),\ \sigma_{xy}^{(III)}(x,1.5\Phi + s) = \sigma_{xy}^{(IV)}(x,1.5\Phi + s) \tag{23a,b}$$

Then, the unkown functions $f_1(x)$, $f_2(x)$, $f_3(x)$, $f_4(x)$ $g_1(x)$, $g_2(x)$, $g_3(x)$, $g_4(x)$ can be determined by the boundary and continuity conditions.

Substituting Eqs.(1)-(4) into Eqs.(20a,b) yields

$$\varphi'(\frac{L}{2}) = 0 \tag{23}$$

$$\psi(\frac{L}{2}) = -1 \tag{24}$$

The influence of the fiber break to the stress distributions on the left surface is not obvious for a high ratio L/Φ . We assume

$$\sigma_{xx}(0,y) = \sigma_{xx_0}(0,y), \sigma_{xy}(0,y) = 0 \tag{25a,b}$$

Substituting Eqs.(10)-(13) into Eqs.(25a,b) yields

$$\varphi'(0) = 0 \tag{26}$$

$$\varphi(0) = 0 \tag{27}$$

For no displacement boundary part, the complementary energy is

$$\begin{aligned}\Pi_c(\sigma_{ij}) &= \int_v W(\sigma_{ij})dV \\ &= \int_0^{L/2}\left[\int_0^{\frac{\Phi}{2}} W_c^{(\mathrm{I})}(\sigma_{ij})dy\right]dx + \int_0^{L/2}\left[\int_{\frac{\Phi}{2}}^{\frac{\Phi}{2}+s} W_c^{(\mathrm{II})}(\sigma_{ij})dy\right]dx \\ &\quad + \int_0^{L/2}\left[\int_{\frac{\Phi}{2}+s}^{\frac{3\Phi}{2}+s} W_c^{(\mathrm{III})}(\sigma_{ij})dy\right]dx + \int_0^{L/2}\left[\int_{\frac{3\Phi}{2}+s}^{h} W_c^{(\mathrm{IV})}(\sigma_{ij})dy\right]dx\end{aligned} \tag{28}$$

Taking the first variation of Eq.(39), we have

$$\delta\Pi_c = \int_{-L/2}^{L/2}\left[a_4\varphi^{(4)}(x) + a_2\varphi''(x) + a_0\varphi(x) + b_0\right]\delta\varphi(x)dx \tag{29}$$

where constants a_4, a_2, a_0 and b_0 are given in Appendix A.

According to the principle of the minimum complementary energy, the total complementary energy $\prod_c$ is required to satisfy the equation

$$\delta\Pi_c(\sigma_{ij}) = \delta\Pi_c\left[\varphi(x)\right] = 0 \tag{30}$$

Applying Eq.(29) to Eq.(30), a fourth-order ordinary differential equation is obtained

$$a_4\varphi^{(4)}(x) + a_2\varphi''(x) + a_0\varphi(x) + b_0 = 0 \tag{31}$$

Eq.(31), together with the boundary conditions listed in Eqs.(23)-(24) and Eqs.(26)-(27), define the boundary value problem for determining $\varphi(x)$. With the function $\varphi(x)$ determined, the stress field and the stress concentration factor can then be obtained. In the present work, the SCF, K, is defined as the ratio between the local stress and the applied stress in the fiber far away from the break site:

$$K = \frac{\sigma_{local}}{\sigma_{applied}} \tag{32}$$

3 Results and discussion

In the following, the results predicted by this variational solution are presented. From the model, the SCF appears to a function of the inter-fiber spacing and the moduli of the fiber and matrix.

Fig. 3 shows the SCF in the break fiber and an intact fiber immediately adjacent to the break fiber with different inter-fiber spacing. The SCF is plotted against position along the fiber. From Fig. 3, it can be seen that an increase of inter-fiber spacing leads to a decrease of both the ineffective length in the break fiber and positively affected length in the adjacent fiber. Fig. 4 shows the influence of the inter-fiber spacing on the maximum SCF. It can be seen that the maximum SCF decreases proportionally from a value of 1.49865 at an inter-fiber spacing of 0.2ϕ to a value of 1.4158 at an inter-fiber spacing of 15ϕ. Similar observations were reported by Grubb et al.[7] and Fukuda et al.[8].

4 Conclusions

- In this work, the variational theory is used to perform micromechanical analyses to determine the SCF of unidirectionally reinforced fiber composite due to a fiber break. Using the variational solution, the stress components are derived in terms of a perturbation function, which is governed by a fourth-order ordinary differential equation. The influences of the inter-fiber spacing, the elastic properties of fiber and matrix to the SCF are investigated. The SCF is found to increase with a decrease of the inter-fiber spacing.

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Appendix A

$$
\begin{aligned}
a_0 = \frac{1}{960}\Big\{ & 96\Phi^5 S^f_{yy}\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 160\Phi^2 s\left(3\Phi^2 + 6\Phi s + 4s^2\right) S^m_{yy}\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 8\Phi^3 S^f_{yy}\big[120 s^2 \\
& - 20\Phi s\left(e_1 - 6\right) + \Phi^2\left(30 - 10e_1 + e_1^2\right)\big]\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 - 80\Phi^3 S^f_{yy}\big[9\Phi^2 + 4s^2 + 26\Phi s \\
& - h^2\big]\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} + 20\Phi^2 S^f_{yy}\big[3\Phi^3\left(e_1 - 6\right) + 12sh^2 - 24\Phi s\left(4s + h\right) + 2\Phi^2\left(s\left(4e_1 - 42\right)\right. \\
& \left. + h\left(e_1 - 6\right)\right)\big]\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} - 80\Phi s S^m_{yy}\big[9\Phi^3 + 3\Phi^2\left(5s + 2h\right) + s\left(-2s^2 + 4sh - 3h^2\right) \\
& + \Phi\left(8s^2 + 12sh - 3h^2\right)\big]\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} - 3S^m_{yy}\left(3\Phi + 2s - h\right)^5\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + 15\Phi S^f_{yy}\big[-3\Phi^2 + h^2 \\
& - 2\Phi\left(4s + h\right)\big]^2\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + 10\Phi S^f_{yy}\big[243\Phi^4 + 1080\Phi^3 s + 2\Phi^2\left(932 s^2 - 27h^2\right) \\
& + 3\left(-12s^2 + h^2\right)^2 + 24\Phi\left(60s^3 - 5sh^2\right)\big]\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + 2sS^m_{yy}\big[135\Phi^4 + 180\Phi^3\left(s + h\right) + \\
& + 30\Phi^2\left(4s^2 + 10sh - h^2\right) + 20\Phi\left(2s^3 - 4s^2 h + 3sh^2 - 3h^3\right) + 3\left(16s^4 - 40s^3 h\right. \\
& \left. + 40s^2h^2 - 20sh^3 + 5h^4\right)\big]\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2\Big\}
\end{aligned}
$$

$$
\begin{aligned}
b_0 = \frac{1}{12}\Big\{ & -24\Phi^2 s S^m_{ss}\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 8\Phi^3 s\left(-S^f_{ss} + S^f_{xy}\right)\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 - \Phi^2 e_1\big[-24sS^f_{xy} \\
& + \Phi\left(2S^f_{xy}\left(e_1 - 6\right)\right) + S^f_{ss}e_1\big]\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 24\Phi s\left(\Phi + s\right)S^m_{xy}\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} - 24\Phi^2 s S^f_{ss}\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} \\
& + 24\Phi s\left(4s - h\right)S^m_{ss}\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} - 6\Phi S^f_{xy}\big[9s^2 + 12s^2 + 20\Phi s - h^2\big]\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} \\
& - 3\Phi S^f_{xy}e_2\big[3s^2 + 2\Phi\left(4s + h\right) - h^2\big]\sigma^{(\mathrm{III})}_{xx_0}\sigma^{(\mathrm{IV})}_{xx_0} - 24\Phi S^f_{ss}s^2\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + \left(S^f_{ss} - S^m_{xy}\right)\left(3\Phi\right. \\
& \left. + 2s - h\right)^3\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 - 2sS^m_{ss}\big[27\Phi^2 + 4s^2 + 18\Phi s - 6\left(3\Phi + s\right)h + 3h^2\big]\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 \\
& + 2sS^m_{xy}\big[-9\Phi^2 + 4s^2 - 6sh + 3h^2 - 6\Phi\left(s + h\right)\big]\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2\Big\}
\end{aligned}
$$

$$c_0 = 2\Phi S^f_{xx}\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + \Phi S^f_{xx}\left(e_1\right)^2\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 2sS^m_{xx}\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + \left(h - 3\Phi - 2s\right)S^f_{xx}\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2$$

$$d_0 = 2\Phi S^f_{xx}\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + \Phi S^f_{xx}e_1\left(\sigma^{(\mathrm{III})}_{xx_0}\right)^2 + 2sS^m_{xx}\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2 + \left(h - 3\Phi - 2s\right)S^f_{xx}\left(\sigma^{(\mathrm{IV})}_{xx_0}\right)^2$$

$$e_1 = -\frac{4sE_m + 2\Phi E_f}{\Phi E_f}$$

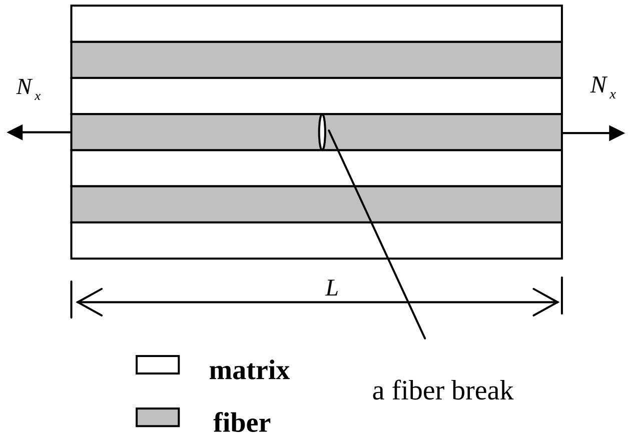


Fig.1 Schematic diagram of fiber reinforced composite with a fiber break

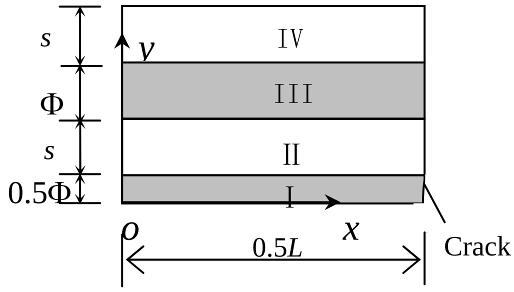
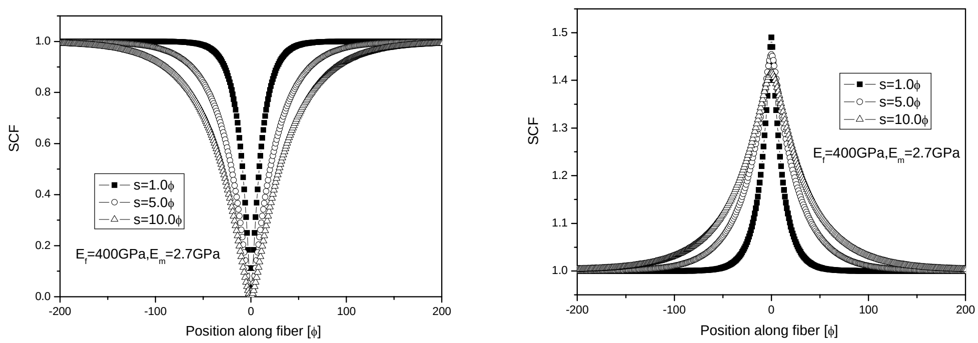


Fig.2 Mechanical model of fiber reinforced composite with a fiber break



(a)

(b)

Fig.3 Influence of the inter-fiber spacing on (a) a break fiber and (b) its adjacent fiber

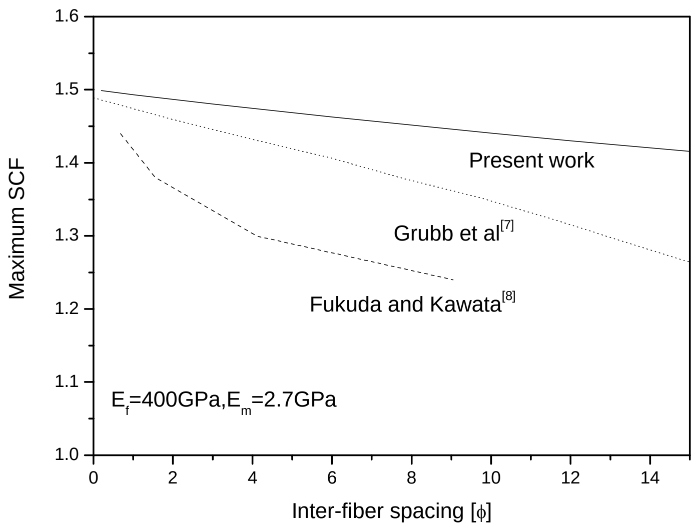


Fig.4 Influence of inter-fiber spacing on the maximum SCF in a fiber adjacent to a break fiber