Explicit Use of the Matrix Yield Condition for Restricting Damage-plasticity in Porous Materials

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Abstract

In the Gurson model, the derivation of a yield function employs the hypothesis of plasticity in matrix material as a primary basis. The obtained condition takes account of both plasticity in matrix and growing of micro-voids embedded in the matrix. Then physical problems are solved by defining damage-plasticity on the associated yield surface. But the porous material contains also the matrix material as a physical component that undergoes its own evolution. Since, in Damage Mechanics, the hypothesis of effective stress relies stresses in matrix and porous material, the matrix yield condition may be written explicitly as a function of stresses in porous material. So a new condition appears to restrict the Gurson condition, leading to some non-smooth yield surface. The present contribution develops this scheme to several generalized Gurson conditions.

1. Introduction: the Gurson model

The Gurson model was first presented in an original paper in 1977 [1], in order to obtain a yield surface for porous rigid perfect-plastic material. When the matrix material surrounding spherical micro-voids is a von Mises material, Gurson proposed an approximate yield surface under the form

$$f(\sigma,x) = (\sigma_{eq}/\sigma_0)^2 + 2x\, cosh(\tfrac{1}{2} tr\sigma/\sigma_0) - 1 - x^2 = 0,$$

(*cosh*: hyperbolic cosine; σ_{eq} : equivalent tensile stress; $tr\sigma$: trace of the Cauchy stress tensor σ). Constant σ_0 is the equivalent tensile yield stress in the matrix material and x is the void volume fraction v ($x=v$, $0 \le v \le 1$). Note that this yield condition gives at complete ductile failure where $\sigma = 0$

$$2x - 1 - x^2 = 0 \quad or \quad x = 1$$

i.e. a limiting value $x_u = 1$ for $x=v$ (evidently, failure arises before this value).

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In subsequent works, the Gurson surface and generalized similar surfaces were used by the authors for write normality law in plastic-porous material considered as a whole without reference to the matrix material. We can refer to references [2,6] for such treatments of the problem. Li [7] was the first to show a link between this Void Growth Model and Continuum Damage Mechanics by introducing the so-called "effective stress" concept [8], [9]. So he recovered a generalized Gurson function by an inverse method, and in the sequel used the classical normality rule with this surface to treat the damage-plasticity evolution.

Here we consider an other point of view and, in a new approach of the problem, we adapt this model to Damage Mechanics by using the "effective stress" concept. In reality, the matrix material is the effective material suffering stresses σ_r related to the Cauchy stresses σ by means of a (linear) operator *M* or (in our case) a scalar function *y*, under the classical forms

$$\sigma = M(D):\sigma_r \quad , \quad \sigma = y(x)\,\sigma_r$$

The fourth-rank operator *M* is said the "damage effect tensor" and depends on damage variables such as the area density of damage *D*. In the second formula, *y(x)* must be a positive scalar (regular) function of some damage-variable *x* related to the micro-voids volume fraction (see Part 2), satisfying to the following (mathematical) properties:

$$dy/dx < 0 \quad , \quad y(0) = 1 \quad , \quad y(x_u) = 0$$

(x_u is the ultimate value of *x*) in order to take account of the damaging process throughout the porous material. Naturally the function *y(x)* is unknown and its determination depends on the mechanical problem under consideration. In the reference [10], this problem was solved in a specific case briefly recalled in Part 2. In subsequent parts, other cases are analysed.

2. The Tvergaard-Needleman model

In this section, we consider a generalized Gurson yield surface for porous rigid perfect-plastic material with spherical micro-voids [11], viz

$$f(\sigma,x) = Q^2 + 2qx\cosh(H/2) - 1 - q^2x^2 = 0, \;\; Q = \sigma_{eq}/\sigma_0\,, \;\; H = tr\,\sigma/\sigma_0$$

where $q=3/2$ and *H, Q* are normalization parameters since σ_0 is a constant. In the papers [4] or [7] by example, *x* is a function of the void volume fraction *v* to take account of void coalescence (after nucleation and growth) before fracture

$$x(v) = v \quad if \;\; 0 \le v \le v_c$$

$$x(v) = v_c + \frac{x_u - v_c}{v_F - v_c}(v - v_c) \quad if \quad v_c \leq v \leq v_F$$

where $v_c=0.15$ is the critical value of v at which void coalescence begins, $v_F=0.25$ its value at complete ductile failure, and x_u the ultimate value of x, obtained from the yield condition when $\sigma = 0$

$$2qx_u-(1+q^2x_u^2)=0 \quad \text{or} \quad x_u=1/q=2/3$$

since at complete failure the material is at yielding even if stresses vanish. So the parameter x increases from 0 to $1/q=2/3$ when the void volume fraction v increases from 0 to $v_F=0.25$ (see Fig.2 in Part 5).

Now, in the framework of Damage Mechanics recalled in Part 1, if we introduce the von Mises surface used for the rigid perfect plastic matrix material, it is seen that this surface may be expressed in function of the Cauchy stress tensor σ in place of the effective stress tensor σ_r , so writing the von Mises yield surface

$$f_r(\sigma ,x)=Q^2-y^2(x)=0$$

So the reversible region is the intersection of the interiors of the yield surfaces $f=0$ and $f_r=0$, i.e. the interiors of the regions of boundary OABC on Fig.1.

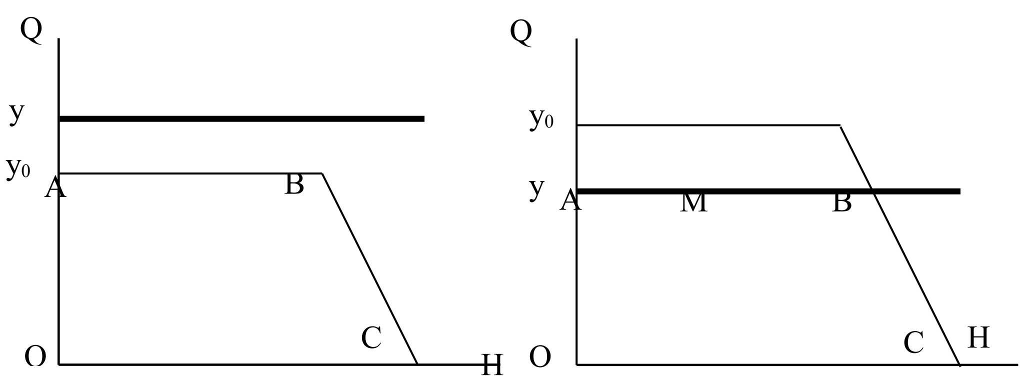

Figure 1. Interiors of OABC (1/4 meridian section) are the accessible stress states ($y_0(x)=1-qx$). Gurson surfaces are approximated by linear segments (thin lines).

But, due to the rigidity of the matrix material, the region defined by $f<0$ ($y_0<y$) cannot be contained in the region defined by $f_r<0$ (Fig1.a). So we have necessarily the stress states associate to Fig.1b ($y_0>y$) , along AMB and on point B only (due to rigidity of the matrix material). Along AMB plasticity without damage takes place and on B both plasticity and damage arise. For $H=0$ (on the Q-axis), we have for the respective values Q' and Q'' of Q obtained from the yield surfaces $f=0$ and $f_r=0$

$$Q'=(1-qx) \quad , \quad Q''=y(x)$$

Therefore it results (Fig.1b) from mechanical considerations

$$y(x) \leq y_0(x) = (1-qx)$$

A process of damage-plasticity arises if the stress point remains on both the two yield surfaces at the non-smooth point B, so obtaining H and Q by the formulae

$$\cosh(H/2) = (1+q^2x^2-y^2(x))/2qx \quad , \quad Q=y(x)$$

Finally we recall the properties that must be satisfied by the unknown function $y(x)$: y is decreasing on the interval $[0,x_u]$, $y(0)=1$, $y(x_u)=0$, $y(x) \leq y_0(x)=1-qx$.

As an example we consider the uniaxial tensile test with ($\sigma_{11}=m\sigma_0>0$), so that $H=m$ and $Q=m$. The point B is defined by

$$2qx\cosh(m/2) = 1+q^2x^2-y^2(x) \quad , \quad m=y(x)$$

These equations give the two unknown quantities m and y. In particular the function $y(x)$ is defined by an implicit relation obtained by eliminating m , viz

$$y^2+2qx\cosh(y/2) -1-q^2x^2=0 \qquad 0 \leq x \leq x_u$$

or

$$y^2+2qx(\cosh(y/2)-1)-(1-qx)^2=0 \qquad 0 \leq x \leq x_u$$

It results

$$y(0)=1 \ , \ y(x_u)=y(1/q)=0 \ , \ y(x) \leq y_0(x) = (1-qx),$$

and by derivation

$$(2y+qx\sinh(y/2))(dy/dx) = -2q(\cosh(y/2)-qx)$$

showing that the derivative of y is negative since the maximum value of qx is $qx_u=1$, and that the function y(x) is decreasing. Finally we can solve x in function of y (y is a monotonic function) so obtaining

$$qx(y) = \cosh(y/2)-(y^2+\sinh^2(y/2))^{1/2} \qquad 0 \leq y \leq 1$$

in order to draw the representative curve of the function $y(x)$. So the problem of the determination of the decreasing function $y(x)$ is achieved. In reference [10], the problem of finite simple shear is also analysed.

3. The Garajeu-Suquet model

In this part, with the notations of Part 1 and 2, we consider the generalized yield surface [12]

$$f(\sigma,x)=(1+2x/3)Q^2+2x\cosh(H/2)-1-x^2=0$$

This relation gives at complete failure (where $\sigma=0$) the ultimate value $x_u=1$. The relative positions of this yield surface and the von Mises surface $f_r=0$ are analogous to the positions described in Fig.1, but here the function $y_0(x)$, (defining the point on the Q-axis) is given by

$$y_0(x)=(1-x)/(1+2x/3)^{1/2}$$

Due to the rigidity of the matrix material, only the Fig. 1b is suitable; so the following mechanical condition must be satisfied

$$y(x)\leq y_0(x)=(1-x)/(1+2x/3)^{1/2}$$

The damage-plasticity arises if the following conditions are fulfilled (point B on Fig.1)

$$2x\cosh(H/2)=(1+x^2-(1+2x/3)\,y^2(x)) \quad , \quad Q=y(x)$$

In this scheme, the function $y(x)$ must satisfied to the properties given in Part 1. Its complete determination depends on the problem under consideration. By example for the uniaxial tensile test (see Part 2), we have the implicit equation

$$(1+2x/3)y^2+2x\cosh(y/2)-1-x^2=0 \qquad 0\leq x\leq 1$$

or

$$y^2+[2x/(1+2x/3)](\cosh(y/2)-1)-y_0^2(x)=0 \qquad 0\leq x\leq 1$$

It results $y(0)=1$, $y(1)=0$, $y(x)\leq y_0(x)$ and by derivation

$$[(1+2x/3)y+(x/2)\sinh(y/2)]\,(dy/dx)=-[(\cosh(y/2)-x)+y^2/3]$$

Since $x=x_u=1$ is the maximum value of x, the second member of this equality is negative and $y(x)$ is decreasing (so a monotonic function of x). Then, in order to draw the graph of the function $y(x)$, it is possible to express x as a function of y, so obtaining

$$x(y)=[\cosh(y/2)+y^2/3]-[(\cosh(y/2)+y^2/3)^2+y^2-1]^{1/2} \quad , \quad 0\leq y\leq 1$$

where the sign (-) was chosen to satisfy the equality $y(0)=1$. So the determination of $y(x)$ is achieved.

4. The Hashin-Shtrikman model

Here we consider the generalized yield surface [13]

$$(1+2x/3)Q^2+x(H/2)^2-(1-x)^2=0$$

At complete failure ($H=0,\ Q=0$) we have the ultimate value $x_u=1$. The relative positions of this yield surface and the von Mises surface are similar to the positions shown in Fig.1. The analogous point on the Q-axis is given by

$$y_0(x)=(1-x)/(1+2x/3)^{1/2} \quad 0\leq x\leq 1$$

and must satisfy the inequality $y(x)\ \leq y_0(x)$. In the case of damage-plasticity (point B on Fig.1), H and Q are given by

$$x(H/2)^2=(1-x)^2-(1+2x/3)\ y^2(x)) \quad , \quad Q=y(x)$$

in function of $y(x)$ that must be determined in each mechanical problem.

For the uniaxial problem of the Part 2, this function is

$$y(x)=(1-x)/(1+11x/12)^{1/2}$$

and satisfies to the necessary conditions, $dy/dx<0,\ y(0)=1\ ,\ y(1)=0\ \ ,\ y(x)\leq y_0(x)\ )$. The determination of y(x) is achieved.

5 The Tvergaard model

As a final example, we consider a generalization of the Gurson model [14] used by several authors. For reason of brevity, only the yield surface is written

$$f(\sigma,x,\sigma_M)=(Q_2/\sigma_M)^2+2q_1x\cosh(\tfrac{1}{2}q_2Q_1/\sigma_M)-1-q_3x^2=0,\ Q_2=\sigma_{eq}\ ,\ \ Q_1=tr\,\sigma$$

(in this part, Q_1 and Q_2 are not normalized stresses) where $q_1,\ q_2,\ q_3$ are fitting constant positive parameters used in order to take account of some specific properties for porous materials. We suppose that the matrix is rigid-plastic with a von Mises criterion and σ_M denotes the flow yield strength of the matrix material (where σ_M is a known function of the equivalent plastic deformation ε_M of the matrix material; by example $\sigma_M=\sigma_0+k\varepsilon_M$, k being a material constant).

The parameter x has the meaning given in Part 2. Here we have drawn the representative curve of x in function of v on Fig.2. We note that the inequalities $x_u > v_F = 0.25 > v_c = 0.15$ are satisfied, so that the linear piece-wise curve is drawn on Fig.2. The ultimate value of x is given by the yield surface when $\sigma = 0$, so obtaining

$$2q_1 x_u - 1 - q_3 x_u^2 = 0$$

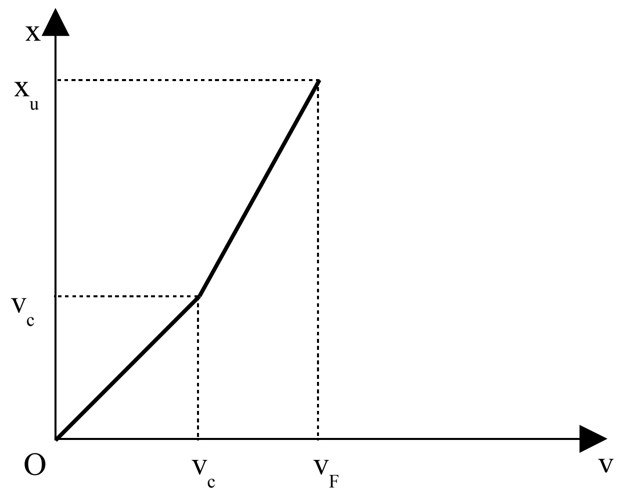

Figure 2.Graph of the function $x(v)$ with $v_F < x_u < q_1/q_3$.

This quadratic equation gives two values of x_u if the existence condition $q_1^2 - q_3 \geq 0$ is satisfied. But, for mechanical reasons, only the expression

$$q_3\, x_u = q_1 - (q_1^2 - q_3)^{1/2}$$

for the limiting value x_u of x will be retained, since it is the first value associated to the complete failure (i.e. $\sigma = 0$). Note that $q_1 - q_3\, x_u \geq 0$. The particular case of Part 2, where $q_3 = q_1^2$, $q_2 = 1$ is relevant of the above condition.

Now the effective stress tensor σ_r is introduced, related to the Cauchy stress tensor σ by using the function $y(x)$ as it was made in section 2, with the same conditions, viz $dy/dx < 0$, $y(0) = 1$, $y(x_u) = 0$. Taking account of the von Mises condition on the matrix material, it results

$$\sigma_{eq} = y(x)(\sigma_r)_{eq} = y(x)\sigma_M$$

so that the yield condition on the porous material becomes

$$f(\sigma, x, \sigma_M) = y^2 + 2q_1\, x \cosh(\tfrac{1}{2} q_2 Q_1 / \sigma_M) - 1 - q_3 x^2 = 0$$

We note that this quadratic form (relatively to x) allows to write the variable x as a function of y and σ, under the form

$$q_3\, x = q_1 cosh(\tfrac{1}{2} q_2 Q_1 / \sigma_M) - [(q_1^2 cosh(\tfrac{1}{2} q_2 Q_1 / \sigma_M) - q_3(1-y^2)]^{1/2}$$

since the term between brackets may be easily seen greater than the positive quantity (q_1^2-q_3) (a condition obtained for the existence of x_u). In writing this expression we have use the fact that $y=1$ for $x=0$, so eliminating the second root of the quadratic equation. This result permits to recover the choice of the value x_u given for the limiting case $x=x_u$ where σ=0 (so $Q_1=0$) and $y(x_u)=0$ (this previous choice was justified by mechanical consideration).

The determination of the function $y(x)$ must be made in each particular problem. We consider the uniaxial tensile test with ($\sigma_{11}=m>0$), so that $Q_1=m$ and $Q_2=m$. In an analogous way as in Part 2, damage-plasticity arises on point B defined by

$$m/\sigma_M = y \quad , \quad m^2/\sigma_M{}^2 + 2q_1 x cosh\,(q_2 m / 2\sigma_M) - 1 - q_3 x^2 = 0$$

and $y(x)$ is defined by the implicit relation

$$y^2 + 2q_1 x\, cosh(\tfrac{1}{2}\, q_2\, y) - 1 - q_3\, x^2 = 0\,, \quad 0 \leq x \leq x_u$$

By derivation we have

$$[2y + q_1 q_2 x\, sinh(q_2 y/2)](dy/dx)/2 = -[q_1 cosh(q_2 y/2) - q_3 x]$$

The brackets on the right hand side satisfies to the inequalities

$$q_1 cosh(q_2 y/2) - q_3 x \geq q_1 cosh(q_2 y/2) - q_3 x_u = q_1[cosh(q_2 y/2) - 1] + (q_1^2 - q_3)^{1/2} > 0$$

since x_u is the maximum value of x. So the derivative dy/dx is negative and the function $y(x)$ is decreasing. Finally we have $y(0)=1$ and $y(x_u)=0$. We note also that the graph of the function $y(x)$ may be obtained by the inverse function $x(y)$ since $y(x)$ is a monotonic function. The result is

$$q_3\, x(y) = q_1 cosh(q_2 y/2) - [q_1^2 ch^2(q_2 y/2) - q_3(1-y^2)]^{1/2} \quad , \quad 0 \leq y \leq 1$$

and if $q_3=q_1^2$, $q_2=1$, we recover the expression of $x(y)$ given in Part 2. Finally we remark that the only condition on the parameters q_1, q_2, q_3 is q_1^2-$q_3 \geq 0$.

6. Conclusion

The objective of this paper was to show the ability of the method initiated in Part 2 to various generalized Gurson conditions. The main result is the necessity to restrict the Gurson yield condition by the von Mises yield condition on the matrix material.

In fact it must be noted that the damage-free matrix material evolves together with the damaged porous material, satisfying to its own yield condition whatever the evolution of the porous material. Also the natural background to introduce the matrix material in an explicit manner is to consider the simultaneous evolutions of two representative material elements, a free-damage element accompanying a damaged element, as it was proposed in the reference [10], by taking account of large strains [15]. Then it becomes evident that the matrix yield condition restricts the porous material yield condition.

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