Elastic Field of an Elliptical Inhomogeneity with Polynomial Eigenstrains in Orthotropic Media

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1. Introduction

Imperfections in microstructures such as voids, cracks and inhomogeneities cause large changes in overall mechanical behavior of the structures. Determination of stress fields inside/outside single and some inhomogeneities is of great importance to understanding of fracture, fatigue strength and failure behavior of the heterogeneous materials. Classical research was reported in the early 1960's by Eshelby [1-3], with many corresponding investigations [4-11]. Asaro and Barnett [12] showed that when the eigenstrain inside an ellipsoidal inclusion is of the form of a polynomial of an arbitrary order in Cartesian coordinates, an induced strain field in the inclusion is also characterized by a polynomial of the same order. The result for the polynomial eigenstrains is referred to as Eshellby's polynomial conservation theorem [13].

The present paper presents an analytic solution for the induced stress field by quadratic distribution of eigenstrains in orthotropic materials with complex roots. Based on principle of minimum potential energy of the elastic inhomogeneity-matrix system, a closed-form solution is obtained by determining the coefficients of some quadratic functions in the coordinates of the points of the inhomogeneity. The results reflect the coupling effect of the zero and second order terms in the polynomial eigenstrains on the elastic field.

2. Fundamental governing equations

For an orthotropic medium in which two in-plane *x*- and *y*- directions coincide with the principal directions of elasticity, the constitutive and geometric relations under plane strain condition are expressed as

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x} = \beta_{11}\,\sigma_x + \beta_{12}\,\sigma_y\,, \\
\varepsilon_y &= \frac{\partial v}{\partial y} = \beta_{12}\,\sigma_x + \beta_{22}\,\sigma_y\,, \\
\gamma_{xy} &= \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \beta_{66}\,\tau_{xy}\,,
\end{aligned} \tag{1}$$

where

$$\beta_{ij} = a_{ij} - \frac{a_{i3}\,a_{j3}}{a_{33}}\,, \quad i, j = 1, 2, 6\,, \tag{2}$$

are elements of reduced compliance matrix for the plane strain condition. Elements of the compliance matrix, $a_{ij} = a_{ji}$, can be determined in terms of the material constants such that

$$a_{11} = \frac{1}{E_1},\ a_{22} = \frac{1}{E_2},\ a_{33} = \frac{1}{E_3},\ a_{12} = a_{21} = \frac{-v_{12}}{E_1} = \frac{-v_{21}}{E_2},$$
$$a_{13} = a_{31} = \frac{-v_{13}}{E_1} = \frac{-v_{31}}{E_3},\ a_{23} = a_{32} = \frac{-v_{23}}{E_2} = \frac{-v_{32}}{E_3},\ a_{66} = \frac{1}{G_{12}}, \tag{3}$$

where E_i , $i = 1,2,3$ is the elastic modulus in the *x*, *y* and *z* directions respectively. $v_{ij}\ (i \neq j\ ;\ j = 1,2,3)$ which is the Poisson's ratio, is the negative of the transverse strain in the *j*-direction over the strain in the *i*-direction when stress is applied in the *i*- direction, and G_{12} is the shear modulus in the *xy*-plane. The stress and displacement components in Eq.(1) can be written by means of two undetermined complex functions $\phi_1(z_1)$, $\phi_2(z_2)$ and their derivatives as follows

$$\begin{aligned} \sigma_x(x,y) &= 2\,\mathrm{Re}\left[\mu_1^2\,\phi_1'(z_1) + \mu_2^2\,\phi_2'(z_2)\right], \\ \sigma_y(x,y) &= 2\,\mathrm{Re}\left[\phi_1'(z_1) + \phi_2'(z_2)\right], \\ \tau_{xy}(x,y) &= -2\,\mathrm{Re}\left[\mu_1\,\phi_1'(z_1) + \mu_2\,\phi_2'(z_2)\right], \end{aligned} \tag{4}$$

and

$$\begin{aligned} u(x,y) &= 2\,\mathrm{Re}[p_1\,\phi_1(z_1) + p_2\,\phi_2(z_2)], \\ v(x,y) &= 2\,\mathrm{Re}[q_1\,\phi_1(z_1) + q_2\,\phi_2(z_2)], \end{aligned} \tag{5}$$

where $z_1 = x + \mu_1 y$ and $z_2 = x + \mu_2 y$. The four complex coefficients p_1, p_2, q_1, q_2 are given by

$$p_1 = \beta_{11}\,\mu_1^2 + \beta_{12}\ ,\ p_2 = \beta_{11}\,\mu_2^2 + \beta_{12},\ q_1 = \beta_{12}\,\mu_1 + \frac{\beta_{22}}{\mu_1},\ q_2 = \beta_{12}\,\mu_2 + \frac{\beta_{22}}{\mu_2}, \tag{6}$$

in which μ_1 , μ_2 are two characteristic parameters of the orthotropic media indicating the degree of anisotropy for either complex or purely imaginary numbers [14]. For plane stress, the corresponding governing equations are obtained directly by replacing the elements β_{ij} with a_{ij} for $i, j = 1,2,6$.

3. Elastic fields and strain energy induced by eigenstrains

3.1 Elastic field inside the inhomogeneity

Consider an elliptic inhomogeneity with non-uniform normal and shear eigenstrains, $\varepsilon_x^*(x,y)$, $\varepsilon_y^*(x,y)$, $\gamma_{xy}^*(x,y)$, embedded in an infinite homogeneous and orthotropic linear elastic solid, the eigenstrains induce elastic strains in the inhomogeneity, denoted by $\varepsilon_x^0(x,y)$, $\varepsilon_y^0(x,y)$, $\gamma_{xy}^0(x,y)$. The total strain components $\varepsilon_1(x,y)$, $\varepsilon_2(x,y)$, $\gamma_{12}(x,y)$ are written as

$$\varepsilon_1 = \varepsilon_x^0 + \varepsilon_x^*,\ \varepsilon_2 = \varepsilon_y^0 + \varepsilon_y^*,\ \gamma_{12} = \gamma_{xy}^0 + \gamma_{xy}^*\ . \tag{7}$$

Specifically, the eigenstrains are assumed to be quadratic in Cartesian coordinates of the points of the inhomogeneity such that

$$\begin{aligned} \varepsilon_x^* &= c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 x y + c_5 y^2 \\ \varepsilon_y^* &= d_0 + d_1 x + d_2 y + d_3 x^2 + d_4 x y + d_5 y^2 \\ \gamma_{xy}^* &= e_0 + e_1 x + e_2 y + e_3 x^2 + e_4 x y + e_5 y^2 \end{aligned} \tag{8}$$

Based on the polynomial conservation theorem [12,13], the total strains can also be expressed in the form of quadratic polynomials as

$$\begin{aligned} \varepsilon_1 &= D_0 + D_1 x + D_2 y + D_3 x^2 + D_4 x y + D_5 y^2 \\ \varepsilon_2 &= E_0 + E_1 x + E_2 y + E_3 x^2 + E_4 x y + E_5 y^2 \\ \gamma_{12} &= F_0 + F_1 x + F_2 y + F_3 x^2 + F_4 x y + F_5 y^2 \end{aligned} \tag{9}$$

where D_i, E_i, F_i, $i = 0,1,\cdots,5$ are 18 real unknown constants determined by the 18 real coefficients c_i, d_i, e_i, $i = 0,1,\cdots,5$ in Eq.(8). Using Eq.(7), the elastic strains in the inhomogeneity are expressed as

$$\begin{aligned} \varepsilon_x^0 &= (D_0 - c_0) + (D_1 - c_1) x + (D_2 - c_2) y + (D_3 - c_3) x^2 + (D_4 - c_4) x y + (D_5 - c_5) y^2 \\ \varepsilon_y^0 &= (E_0 - d_0) + (E_1 - d_1) x + (E_2 - d_2) y + (E_3 - d_3) x^2 + (E_4 - d_4) x y + (E_5 - d_5) y^2 \\ \gamma_{xy}^0 &= (F_0 - e_0) + (F_1 - e_1) x + (F_2 - e_2) y + (F_3 - e_3) x^2 + (F_4 - e_4) x y + (F_5 - e_5) y^2 \end{aligned} \tag{10}$$

From Eq.(1), the stress components in the inhomogeneity under plane strain condition can thus be derived as

$$\begin{pmatrix} \sigma_x^0 \\ \sigma_y^0 \\ \tau_{xy}^0 \end{pmatrix} = \frac{1}{\beta_{11}^0 \beta_{22}^0 - (\beta_{12}^0)^2} \begin{bmatrix} \beta_{22}^0 & -\beta_{12}^0 & 0 \\ -\beta_{12}^0 & \beta_{11}^0 & 0 \\ 0 & 0 & \beta_{11}^0 \beta_{22}^0 - (\beta_{12}^0)^2 \big/ \beta_{66}^0 \end{bmatrix} \begin{pmatrix} \varepsilon_x^0 \\ \varepsilon_y^0 \\ \gamma_{xy}^0 \end{pmatrix} \tag{11}$$

To distinguish between the inhomgeneity from the matrix, the material constants and elastic fields are denoted with the superscript 0 and elastic strain energy for the inhomogeneity are obtained as

$$W_I = \frac{1}{2} \iint_{\Omega} (\varepsilon_x^0 \sigma_x^0 + \varepsilon_y^0 \sigma_y^0 + \gamma_{xy}^0 \tau_{xy}^0) \, dx \, dy , \tag{12}$$

where Ω represents the elliptic region. Substitution of Eqs.(10) and (11) into Eq.(12) yields

$$W_I = \frac{1}{2} \pi a b (A_0 + \frac{1}{4} a^2 A_1 + \frac{1}{4} b^2 A_2 + \frac{1}{8} a^4 A_3 + \frac{1}{24} a^2 b^2 A_4 + \frac{1}{8} b^4 A_5), \tag{13}$$

in which A_i , $i = 0,1,...,5$ are coefficients concerning the known coefficients D_i, E_i, F_i, $i = 0,1,\cdots,5$. As displacements in the inhomogeneity u^0 and v^0 are compatible with the total strains ε_1, ε_2 and γ_{12}, they can be determined as

$$\varepsilon_1 = \frac{\partial u^0}{\partial x}, \ \varepsilon_2 = \frac{\partial v^0}{\partial y}, \ \gamma_{12} = \frac{\partial u^0}{\partial y} + \frac{\partial v^0}{\partial x} . \tag{14}$$

By means of Eqs.(9) and (14), the displacement components are expressed as

$$u^0 = D_0 x + I_1 y + \frac{1}{2} D_1 x^2 + D_2 x y + \frac{1}{2}(F_2 - E_1) y^2 + \frac{1}{3} D_3 x^3 + \frac{1}{2} D_4 x^2 y + D_5 x y^2 + \frac{1}{3}(F_5 - \frac{1}{2} E_4) y^3 + \hat{c}_1 ,$$

$$v^0 = E_0 y + I_2 x + E_1 x y + \frac{1}{2} E_2 y^2 + \frac{1}{2}(F_1 - D_2) x^2 + E_3 x^2 y + \frac{1}{2} E_4 x y^2 + \frac{1}{3} E_5 y^3 + \frac{1}{3}(F_3 - \frac{1}{2} D_4) x^3 + \hat{c}_2 , \quad (15)$$

and

$$F_4 = 2 D_5 + 2 E_3 , \; I_2 = F_0 - I_1 \quad (16)$$

and $\hat{c}_1$ and $\hat{c}_2$ are two real constants without any effect on the stress components. The above resulting displacement components consist of the 18 unknown independent constants D_0, E_0, , I_1, I_2, D_1, E_1, D_2, E_2, F_1, F_2, D_3, E_3, D_4, E_4, D_5, E_5, F_3, F_5 to be determined.

3.2 Strain energy for the matrix

For an elliptic inhomogeneity, the plane region outside the elliptic region can be transformed into a unit circle $|\zeta| < 1$ using the mapping function $z = \omega(\zeta) = (a+b)/(2\zeta) + (a-b)\zeta/2$. On the boundary of the unit circle, $\zeta = \sigma = e^{i\theta}$. Due to the preceding mapping function, the real and imaginary parts of $z = x + i y$ are

$$x = a\cos\theta = \frac{a}{2}\left(\sigma + \frac{1}{\sigma}\right), \; y = -b\sin\theta = -\frac{b}{2i}\left(\sigma - \frac{1}{\sigma}\right), \quad (17)$$

respectively. By means of the continuity condition for displacement at the interface between the inhomogeneity and matrix, $u = u^0$, $v = v^0$, combining Eqs.(15) with (17), the two displacement components at the interior boundary of the matrix can be expressed in terms of σ as

$$u(\sigma) = K_0 + K_1 \sigma + K_2 \sigma^2 + K_3 \sigma^3 + K_4 \frac{1}{\sigma} + K_5 \frac{1}{\sigma^2} + K_6 \frac{1}{\sigma^3},$$

$$v(\sigma) = L_0 + L_1 \sigma + L_2 \sigma^2 + L_3 \sigma^3 + L_4 \frac{1}{\sigma} + L_5 \frac{1}{\sigma^2} + L_6 \frac{1}{\sigma^3}, \quad (18)$$

where K_i, L_i , $i = 0,1,...,6$ are coefficients concerning the above unknown constants. In addition, Eq.(5) can be rewritten as

$$2\,\mathrm{Re}[\, p_1 A(\sigma) + p_2 B(\sigma)] = u(\sigma), \; 2\,\mathrm{Re}[q_1 A(\sigma) + q_2 B(\sigma)] = v(\sigma), \quad (19)$$

where $A(\sigma)$ and $B(\sigma)$ are two transformed equivalent quantities of $\phi_1(z_1)$ and $\phi_2(z_2)$ respectively. The functions $A(\zeta)$ and $B(\zeta)$ can be determined using Schwartz formula [15] as

$$X(\zeta) = \frac{1}{2\pi i}\int_{\Gamma} Y(\sigma)\frac{\sigma + \zeta}{\sigma - \zeta}\frac{d\sigma}{\sigma} + i l_0, \quad (20)$$

where the function $X(\zeta)$ is holomorphic inside a unit circle Γ, and $Y(\sigma)$ is the real part of $X(\zeta)$ on the contour of the unit circle, and l_0 is a real constant. Using Eqs.(18) and (19), and applying Schwartz formula (20), together with Cauchy's formula, one yields

$$p_1 A(\zeta)+p_2 B(\zeta)=\Omega_1(\zeta)+i l_1,\ q_1 A(\zeta)+q_2 B(\zeta)=\Omega_2(\zeta)+i l_2$$

where l_1, l_2 are two real constants, and

$$\Omega_1(\zeta)=\frac{1}{2}K_0+K_1\zeta+K_2\zeta^2+K_3\zeta^3,\ \Omega_2(\zeta)=\frac{1}{2}L_0+L_1\zeta+L_2\zeta^2+L_3\zeta^3 \quad (21)$$

The corresponding expressions for $A(\zeta)$ and $B(\zeta)$ are thus derived as

$$A(\zeta)=\frac{1}{(p_1 q_2-p_2 q_1)}\left[q_2\Omega_1(\zeta)-p_2\Omega_2(\zeta)\right]+i c_1',$$

$$B(\zeta)=\frac{1}{(p_1 q_2-p_2 q_1)}\left[-q_1\Omega_1(\zeta)+p_1\Omega_2(\zeta)\right]+i c_2', \quad (22)$$

in which two constants c_1', c_2' have no influence on the stress components and can be neglected. Consider the case for two complex roots $\mu_1=\alpha+i\beta$ and $\mu_2=-\alpha+i\beta$ $(\alpha>0,\beta>0)$, expressions for the four constants p_1, p_2, q_1, q_2 in Eq.(6) becomes [16]

$$p_1=c_{11}+i c_{12}\alpha,\ p_2=c_{11}-i c_{12}\alpha,\ q_1=c_{21}\alpha+i c_{22},\ q_2=-c_{21}\alpha+i c_{22}. \quad (23)$$

where

$$c_{11}=\beta_{12}+(\alpha^2-\beta^2)\beta_{11},\ c_{12}=2\beta\beta_{11},$$

$$c_{21}=\beta_{12}+\frac{\beta_{22}}{\alpha^2+\beta^2},\ c_{22}=\beta\left(\beta_{12}-\frac{\beta_{22}}{\alpha^2+\beta^2}\right), \quad (24)$$

Due to Eq.(22), $A(\sigma)$ and $B(\sigma)$ can be obtained as

$$A(\sigma)=-\frac{1}{2e\alpha}[\Omega_1(\sigma)q_2-\Omega_2(\sigma)p_2],$$

$$B(\sigma)=-\frac{1}{2e\alpha}[-\Omega_1(\sigma)q_1+\Omega_2(\sigma)p_1], \quad (25)$$

in which coefficient e is expressed as

$$e=c_{11}c_{21}+c_{12}c_{22}=\beta_{12}^2+2(\alpha^2+\beta^2)\beta_{11}\beta_{12}+\frac{\alpha^2-3\beta^2}{\alpha^2+\beta^2}\beta_{11}\beta_{22}. \quad (26)$$

Using Eq.(25), stress components σ_x^c, σ_y^c and τ_{xy}^c in the orthotropic matrix on the interior boundary can finally be obtained from Eq.(4) as

$$\begin{pmatrix}\sigma_x^c\\ \sigma_y^c\\ \tau_{xy}^c\end{pmatrix}=\frac{1}{-2eF_1(\theta)F_2(\theta)}\left[M(\theta)\right]_{3\times18}\{\Sigma\}, \quad (27)$$

where

$$\{\Sigma\}=\{D_0,E_0,I_1,I_2,D_1,E_1,D_2,E_2,F_1,F_2,D_3,E_3,D_4,E_4,D_5,E_5,F_3,F_5\}^T \quad (28)$$

and

$$\left[M(\theta)\right]_{3\times 18}=\begin{bmatrix} P_1(\theta) & P_2(\theta) & \ldots\ldots & P_{18}(\theta) \\ Q_1(\theta) & Q_2(\theta) & \ldots\ldots & Q_{18}(\theta) \\ T_1(\theta) & T_2(\theta) & \ldots\ldots & T_{18}(\theta) \end{bmatrix} \tag{29}$$

in which functions $P_i(\theta), Q_i(\theta)$ and $T_i(\theta)$, $i=1,2,...,18$ are omitted. According to Clapeyron's theorem, the strain energy in the matrix can be calculated by

$$W_M=\frac{1}{2}\oint_{\Gamma}(p_{nx}^c u+p_{ny}^c v)\,ds\,, \tag{30}$$

where p_{nx}^c , p_{ny}^c are tractions on the interior boundary of the matrix (i.e., the unit circle Γ) such that

$$\begin{pmatrix} p_{nx}^c \\ p_{ny}^c \end{pmatrix}=\begin{pmatrix} \sigma_x^c & \tau_{xy}^c \\ \tau_{xy}^c & \sigma_y^c \end{pmatrix}\begin{pmatrix} \cos(x,n) \\ \cos(y,n) \end{pmatrix}, \tag{31}$$

in which $\cos(x,n)=dy/ds$, $\cos(y,n)=-dx/ds$ and n is the outward unit normal. Substituting Eq.(31) into Eq.(30), applying resulting Eq.(27) together with Eqs.(17) and (18), noting that $\sigma=e^{i\theta}$, integration with respect to θ results in analytical expressions for the strain energy for the matrix as

$$\begin{aligned} W_M = {} & n_{11} D_0 D_0 + n_{12} D_0 E_0 + n_{111} D_0 D_3 + n_{112} D_0 E_3 + n_{115} D_0 D_5 + n_{116} D_0 E_5 \\ & + n_{22} E_0 E_0 + n_{211} E_0 D_3 + n_{212} E_0 E_3 + n_{215} E_0 D_5 + n_{216} E_0 E_5 + n_{1111} D_3 D_3 \\ & + n_{1112} D_3 E_3 + n_{1116} D_3 E_5 + n_{1212} E_3 E_3 + n_{1215} E_3 D_5 + n_{1515} D_5 D_5 \\ & + n_{1516} D_5 E_5 + n_{1616} E_5 E_5 \\ & + n_{33} I_1 I_1 + n_{34} I_1 I_2 + n_{313} I_1 D_4 + n_{314} I_1 E_4 + n_{317} I_1 F_3 + n_{318} I_1 F_5 \\ & + n_{44} I_2 I_2 + n_{413} I_2 D_4 + n_{414} I_2 E_4 + n_{417} I_2 F_3 + n_{418} I_2 F_5 + n_{1313} D_4 D_4 \\ & + n_{1314} D_4 E_4 + n_{1317} D_4 F_3 + n_{1318} D_4 F_5 + n_{1414} E_4 E_4 + n_{1417} E_4 F_3 \\ & + n_{1418} E_4 F_5 + n_{1717} F_3 F_3 + n_{1718} F_3 F_5 + n_{1818} F_5 F_5 \\ & + n_{55} D_1 D_1 + n_{56} D_1 E_1 + n_{510} D_1 F_2 + n_{66} E_1 E_1 + n_{610} E_1 F_2 + n_{1010} F_2 F_2 \\ & + n_{77} D_2 D_2 + n_{78} D_2 E_2 + n_{79} D_2 F_1 + n_{88} E_2 E_2 + n_{89} E_2 F_1 + n_{99} F_1 F_1 \end{aligned} \tag{32}$$

where the coefficients n_{ij} are again omitted.

4. Determination of unknown coefficients

Total strain energy of the elastic system, consisting of inhomogeneity and matrix is thus $W=W_I+W_M$ where W_I and W_M are expressed in Eq.(13) and (32), respectively. Based on the principle of minimum potential energy, the 18 independent unknown coefficients can be determined by solving separately the following four sets of equations

$$
\begin{cases}
\dfrac{\partial W}{\partial D_0} = 0 \\
\dfrac{\partial W}{\partial E_0} = 0 \\
\dfrac{\partial W}{\partial D_3} = 0 \\
\dfrac{\partial W}{\partial E_3} = 0 \\
\dfrac{\partial W}{\partial D_5} = 0 \\
\dfrac{\partial W}{\partial E_5} = 0
\end{cases}, \quad
\begin{cases}
\dfrac{\partial W}{\partial I_1} = 0 \\
\dfrac{\partial W}{\partial I_2} = 0 \\
\dfrac{\partial W}{\partial D_4} = 0 \\
\dfrac{\partial W}{\partial E_4} = 0 \\
\dfrac{\partial W}{\partial F_3} = 0 \\
\dfrac{\partial W}{\partial F_5} = 0
\end{cases}, \quad
\begin{cases}
\dfrac{\partial W}{\partial D_1} = 0 \\
\dfrac{\partial W}{\partial E_1} = 0 \\
\dfrac{\partial W}{\partial F_2} = 0
\end{cases}, \quad
\begin{cases}
\dfrac{\partial W}{\partial D_2} = 0 \\
\dfrac{\partial W}{\partial E_2} = 0 \\
\dfrac{\partial W}{\partial F_1} = 0
\end{cases}, \qquad (33)
$$

The first two sets of equations in Eq.(33) result in the coefficients concerning zero and second order terms of x and y in Eq.(9) or Eq.(10) as

$$
\begin{Bmatrix} D_0 \\ E_0 \\ D_3 \\ E_3 \\ D_5 \\ E_5 \end{Bmatrix} = \left[A^1\right]^{-1}_{6\times 6} \left[B^1\right]_{6\times 7} \begin{Bmatrix} c_0 \\ d_0 \\ c_3 \\ d_3 \\ c_5 \\ d_5 \\ e_4 \end{Bmatrix}, \quad \begin{Bmatrix} I_1 \\ I_2 \\ D_4 \\ E_4 \\ F_3 \\ F_5 \end{Bmatrix} = \left[A^2\right]^{-1}_{6\times 6} \left[B^2\right]_{6\times 5} \begin{Bmatrix} e_0 \\ c_4 \\ d_4 \\ e_3 \\ e_5 \end{Bmatrix}. \qquad (34)
$$

The last two sets of equations result in the coefficients concerning first order terms of x and y in Eq.(9) or Eq.(10) as

$$
\begin{Bmatrix} D_1 \\ E_1 \\ F_2 \end{Bmatrix} = \left[A^3\right]^{-1}_{3\times 3} \left[B^3\right]_{3\times 3} \begin{Bmatrix} c_1 \\ d_1 \\ e_2 \end{Bmatrix}, \quad \begin{Bmatrix} D_2 \\ E_2 \\ F_1 \end{Bmatrix} = \left[A^4\right]^{-1}_{3\times 3} \left[B^4\right]_{3\times 3} \begin{Bmatrix} c_2 \\ d_2 \\ e_1 \end{Bmatrix}, \quad (35)
$$

where specific expressions for the eight matrices $\left[A^1\right]_{6\times 6}$, $\left[B^1\right]_{6\times 7}$, $\left[A^2\right]_{6\times 6}$, $\left[B^2\right]_{6\times 5}$, $\left[A^3\right]_{3\times 3}$, $\left[B^3\right]_{3\times 3}$, $\left[A^4\right]_{3\times 3}$, $\left[B^4\right]_{3\times 3}$ are omitted here. The remaining coefficient, F_4, concerning the term with respect to xy in the shear strain can be derived using Eq.(16). Applying the resulting analytic expressions for the 18 independent coefficients in Eqs.(34) and (35), the displacement, elastic strain and stress components inside the inhomogeneity thus have their explicit results given by Eq.(15), (10) and (11), respectively.

5. Results of special cases

The preceding resulting solutions for 18 independent coefficients can be divided into two groups, one is referred to both zero-order terms and quadratic terms, and the other corresponds to first-order terms in Eqs.(9) and (10). The two resulting relations in Eq.(34) reveal that even though there are no zero-order term in the prescribed eigenstrains Eq.(8), i.e., $c_0 = d_0 = e_0 = 0$, the quadratic terms in Eq.(8) can cause the zero-order elastic strain components D_0, E_0 and F_0 in Eq.(9), which reflects the coupling effect of the zero and second order terms in the polynomial expression on the elastic fields. In contrast, the first-order terms in the eigenstrains only produce corresponding elastic fields in the form of the first-order terms, as expressed in Eq.(35), which is similar to results of Nie et al. [16].

For the special case of uniform or constant eigenstrains in isotropic media $(\alpha = 0\,,\ \beta = 1)$, $c_i = d_i = e_i = 0$, $i = 1,2,...,5$ and $D_i = E_i = F_i = 0$, $i = 1,2,...,5$ in Eqs.(8) and (9) respectively. The strain energy for the matrix in Eq.(32) can be expressed by the reduced coefficients, D_0 and E_0 and and I_1, I_2, $F_0 = I_1 + I_2$ as $W_M = W_M^1 + W_M^2$, where W_M^1 and W_M^2 represent two parts of the strain energy associated with normal and shear strains respectively, such that

$$W_M^1 = \frac{\pi G a^2}{2\kappa}\left[\varepsilon_1^2(\kappa+1) + 2R(\kappa-1)\varepsilon_1\varepsilon_2 + R^2\varepsilon_2^2(\kappa+1)\right] \quad (36)$$

$$W_M^2 = \frac{\pi G a^2}{2\kappa}[R^2\gamma_1^2(\kappa+1) + 2R\gamma_1\gamma_2(1-\kappa) + \gamma_2^2(\kappa+1)] \quad (37)$$

in which $\varepsilon_1 = D_0$, $\varepsilon_2 = E_0$, $\gamma_1 = I_1$, $\gamma_2 = I_2$ and $G = E/\left[2(1+\nu)\right]$, $\kappa = 3-4\nu$. The above results are the same as those by Jaswon and Bhargava [17], Bhargava and Radhakrishna [18], Willis [19] and Yang and Chou [20].

6. Stresses at the interface between inhomogeneity and matrix

The normal and shear stresses in the inhomogeneity and matrix are calculated by Eq.(11) and Eq.(27) respectively. For any point at the interface, using the transformation formulae [16], the normal stresses $(\sigma_n^c, \sigma_n^{c0})$ and shear stresses (τ_n^c, τ_n^{c0}) at the interface between the inhomogeneity and matrix can be evaluated independently. In the following numerical examples, two materials are considered.

Firstly, for the case when both the inhomogeneity and matrix are isotropic, the material constants are chosen as $E^0 = E = 10.0\,GPa$, $\nu^0 = \nu = 0.50$ and the two parameters in $\mu_1 = \alpha + i\beta$ and $\mu_2 = -\alpha + i\beta$ can be determined as $\alpha = 0, \beta = 1$. For different ratios of $R = b/a$, computational results for the stresses in GPa at a characteristic point $(a, 0)$ at the tip of the ellipse are:

(1) For $R = 1$

$$\sigma_n^c = \sigma_n^{c0} = -3.75c_0 - 1.5625c_3 + 0.3125c_5$$
$$-1.25d_0 - 0.9375d_3 - 0.3125d_5 + 0.3125e_4$$

$$\tau_{ns}^{c} = \tau_{ns}^{c0} = 0.734629 c_4 + 1.05108 d_4 - 1.25 e_0 - 0.928458 e_3 - 0.232256 e_5$$

(2) For $R = 0.5$

$$\sigma_{n}^{c} = \sigma_{n}^{c0} = -5.5555 c_0 - 2.77778 c_3 - 0.0462963 c_5$$
$$- 1.1111 d_0 - 1.2963 d_3 - 0.0462963 d_5 + 0.37037 e_4$$
$$\tau_{ns}^{c} = \tau_{ns}^{c0} = 0.506963 c_4 + 0.534309 \, d_4 - 1.11111 \, e_0 - 1.23459 \, e_3 - 0.0367601 e_5$$

Results show that when the ellipse becomes shallower, the normal stress becomes larger in magnitude with the uniform eigenstrain c_0 but smaller with the quadratic terms in eigenstrains. However, all terms in eigenstrains cause lower shear stresses for a narrower ellipse.

For the case when the matrix is orthotropic and the inhomogeneity is isotropic, the material constants are such that $E^0 = 10.0 GPa$, $\nu^0 = 0.25$, and $E_1 = 0.16 GPa$, $E_2 = 3.24 GPa$, $\nu_{12} = 0.33$, $G_{12} = 1.875 GPa$. The two parameters can be determined as $\alpha = 0.167$ and $\beta = 1.167$. For $R = 1$, there is

$$\sigma_{n}^{c} = \sigma_{n}^{c0} = -2.4239 c_0 - 1.0476 c_3 + 0.3480 c_5$$
$$- 0.7598 d_0 - 0.7715 d_3 - 0.1370 d_5 + 0.1854 e_4$$
$$\tau_{ns}^{c} = \tau_{ns}^{c0} = 0.2651 c_4 + 0.5624 d_4 - 0.6758 e_0 - 0.5245 e_3 - 0.2272 e_5$$

The above results show that the continuity conditions for the normal and shear stresses at the interface between the inhomogeneity and matrix are satisfied.

7. Conclusions

A closed-form solution for elastic field of an elliptic inhomogeneity with quadratic polynomial eigenstrains in orthotropic media is formulated. The elastic energy of the inhomogeneity/matrix system is expressed in terms of 18 unknown real coefficients, which are analytically evaluated by means of the principle of minimum potential energy. The corresponding elastic field in the inhomogeneity is obtained. The resulting stress field in the inhomogeneity is verified using the continuity conditions for the normal and shear stresses at the interface between the inhomogeneity and matrix. The resulting solution reflects the coupling effect of the zero and second order terms in the polynomial expressions on the elastic field. The present analytic solution reduces to known results for some special cases.

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