A Modified Weibull Stress Model for Cleavage Fracture that Incorporates Threshold Toughness

T.L. Anderson[1], B.D. Rose[1],
[1]*Quest Reliability, Boulder, Colorado, USA*

1 BACKGROUND

The Weibull stress model and variations thereof have been used extensively to predict the probability of cleavage fracture in steels in the ductile-brittle transition region [1-7]. The standard Weibull stress model is based on a weakest-link assumption. While the weakest-link model characterizes cleavage fracture *initiation*, it does not account for the possibility of crack arrest in the presence of a steep stress gradient, such as occurs in front of a crack. Cleavage may initiate at a trigger near the crack tip and subsequently arrest because the crack driving force is insufficient for propagation. This results in a minimum toughness, K_{min}, below which cleavage propagation is not possible.

Previous researchers [1-4] have attempted to incorporate minimum toughness into the Weibull stress model, but their approach is inconsistent with the true physical meaning of K_{min}. The present authors have developed a more rigorous formulation that introduces K_{min} into the Weibull stress model through a hazard function.

2 WEAKEST-LINK CLEAVAGE

The initiation of cleavage fracture in carbon and low-alloy steels occurs by a weakest-link mechanism when the material is in the ductile-brittle transition region. Cleavage initiates in a single fracture-triggering particle or other microstructural feature. The probability of cleavage fracture in a given material is a function of applied stress and sample volume. The mathematical underpinnings of weakest link fracture are outlined below, along with the definition of Weibull stress.

Consider a solid with volume V under a uniform applied stress. The probability that this volume has at least one critical cleavage trigger is given by the Poisson distribution:

$$F = 1 - \exp\left(-\rho V\right) \tag{1}$$

Where ρ is the density (number per unit volume) of critical cleavage triggers. Assuming ρ is governed by normal stress, then ρ is a function of the maximum principal stress, σ_1, in the case of a 3D stress field. If the stress field is non-uniform over volume V, Eq. (1) can be generalized as follows:

$$F(\sigma_1) = 1 - \exp\left(-\int_V \rho(\sigma_1)\, dV\right) \tag{2}$$

2.1 Weibull Stress Model

Let us assume that $\rho(\sigma_1)$ follows a power law in stress:

$$\rho(\sigma_1) = \frac{1}{V_o}\left(\frac{\sigma_1}{\sigma_u}\right)^m \tag{3}$$

where m and σ_u are material constants that characterize the density of cleavage triggers, and V_o is a reference volume. For a uniformly stressed volume, substituting Eq. (3) into Eq. (1) gives:

$$F(\sigma_1) = 1 - \exp\left[-\left(\frac{\sigma_1}{\sigma_u}\right)^m \frac{V}{V_o}\right] \tag{4}$$

which is a two-parameter Weibull distribution. For a non-uniform stress distribution, Eq. (4) becomes:

$$F(\sigma_1) = 1 - \exp\left[-\frac{1}{V_o}\int_V \left(\frac{\sigma_1}{\sigma_u}\right)^m dV\right] \tag{5}$$

Let us now define the *Weibull stress* as an equivalent uniform stress acting over volume V such that the cleavage probability is given by

$$F(\sigma_w) = 1 - \exp\left[-\left(\frac{\sigma_w}{\sigma_u}\right)^m\right] \tag{6}$$

Equating the probabilities in Eqs. (5) and (6) leads to

$$\sigma_w = \left[\frac{1}{V_o}\int_V \sigma_1^{\,m} dV\right]^{1/m} \tag{7}$$

The Weibull stress can be viewed as a weighted average stress that depends on the σ_1 distribution and the size of volume V.

We can introduce a threshold fracture stress, σ_{min}, into the assumed functional relationship for ρ:

$$\rho(\sigma_1) = \begin{cases} \dfrac{1}{V_o}\left(\dfrac{\sigma_1 - \sigma_{\min}}{\sigma_u - \sigma_{\min}}\right)^m & \sigma_1 > \sigma_{\min} \\ 0 & \sigma_1 \leq \sigma_{\min} \end{cases} \tag{8}$$

The threshold stress represents the minimum stress at which cleavage is possible, irrespective of the sample volume. Substituting the above definition into Eq. (1) results in a three-parameter Weibull distribution for a uniformly stressed volume:

$$F(\sigma_1) = 1 - \exp\left[-\left(\frac{\sigma_1 - \sigma_{\min}}{\sigma_u - \sigma_{\min}}\right)^m \frac{V}{V_o}\right] \tag{9}$$

For a non-uniform stress field, Eq. (9) is generalized to a volume integral:

$$F(\sigma_1) = 1 - \exp\left[-\frac{1}{V_o}\int_{V^*}\left(\frac{\sigma_1 - \sigma_{\min}}{\sigma_u - \sigma_{\min}}\right)^m dV\right] \tag{10}$$

where V^* includes only the volume over which $\sigma_1 > \sigma_{\min}$. The three-parameter version of Eq. (6) can be defined as follows:

$$F = 1 - \exp\left[-\left(\frac{\hat{\sigma}_w}{\hat{\sigma}_u}\right)^m\right] \tag{11}$$

where

$$\hat{\sigma}_w = \sigma_w - \sigma_{\min} \tag{12a}$$

$$\hat{\sigma}_u = \sigma_u - \sigma_{\min} \tag{12b}$$

Solving for Weibull stress gives

$$\sigma_w = \left[\frac{1}{V_o}\int_{V^*}(\sigma_1 - \sigma_{\min})^m \, dV\right]^{1/m} + \sigma_{\min} \tag{13}$$

Thus the quantity $(\sigma_1 - \sigma_{\min})^m$ is integrated over V^*. Equations (7) and (13) coincide only if $\sigma_{\min} = 0$ or $m = 1$.

2.2 Weakest Link Model for Toughness

In the absence of constraint loss, the crack tip stress field in front of a crack is uniquely related to the applied J-integral. Applying dimensional analysis to this situation results in a functional relationship of the following form:

$$\frac{\sigma_1}{\sigma_o} = f\left(\frac{J}{r\sigma_o}, \theta\right) \tag{14}$$

Where r the radial distance from the crack tip, θ is the angle from the crack plane, and σ_0 is the yield stress. By substituting this function into Eq. (2), it can be shown that the cleavage probability and the crack driving force are related through a two-parameter Weibull distribution with a known exponent:

$$F = 1 - \exp\left[-\left(\frac{J}{\hat{J}_o}\right)^2\right] \tag{15}$$

Note that there is only one fitting parameter, $\hat{J}_o$. Invoking the relationship between K and J for small-scale yielding gives

$$F = 1 - \exp\left[-\left(\frac{K_J}{\hat{K}_o}\right)^4\right] \tag{16}$$

These relationships hold irrespective of the functional form of $\rho(\sigma_1)$. A comparison of Eqs. (11) and (16) implies the following relationship between Weibull stress and stress intensity factor:

$$\hat{\sigma}_w = \hat{\sigma}_u \left(\frac{K_J}{\hat{K}_o}\right)^{4/m} \tag{17}$$

Equation (17) can be expressed in the following form:

$$\hat{\sigma}_w^{\,m} = \mathbb{C} K_J^{\,4} \tag{18}$$

where $\mathbb{C}$ is a constant that incorporates the parameters in Eq. (17). Under large-scale yielding conditions, where constraint loss is possible, the relationship between Weibull stress and crack driving force can be expressed as follows:

$$\hat{\sigma}_w^{\,m} = \mathbb{C} K_J^{\,4} g(J) \tag{19}$$

Where $g(J)$ is a dimensionless constraint parameter that varies with deformation. Under small-scale yielding conditions, $g(J)=1,$ but this factor falls below unity when constraint loss occurs.

3 INCORPORATING THRESHOLD TOUGHNESS

Cleavage initiation is a necessary but not sufficient condition for fracture. Macroscopic crack propagation requires that the crack driving force exceed the arrest toughness. Therefore, fracture toughness cannot fall below a threshold value, $K_{\min}$.

Previous researchers have attempted to introduce $K_{\min}$ to the Weibull stress model by linking it to $\sigma_{\min}$ [1-4]. However, these two threshold quantities are unrelated, so a direct link is not justified. The threshold stress corresponds to the normal stress value below which cleavage initiation is impossible, irrespective of sample volume. The threshold toughness represents the minimum crack driving force for propagation of cleavage fracture. Even if cleavage initiation occurs at the microscopic level due to stresses in excess of $\sigma_{\min}$ at the crack tip, the crack will not propagate macroscopically unless the crack driving force exceeds $K_{\min}$.

The derivation outlined below modifies the Weibull stress model by introducing a conditional probability of propagation through a *hazard function*. Given a random variable *x*, the probability that $X_o \le x \le X$ is given by

$$F = 1 - \exp\left[-\int_{X_o}^{X} H(x)\,dx \right] \tag{20}$$

where $H(x)$ is the hazard function. By comparing Eqs. (16) and (20), we see that the hazard function for weakest-link cleavage under small-scale yielding conditions is given by

$$H(K_J) = \frac{4K_J^{\;3}}{\hat{K}_o^{\;4}} \tag{21}$$

Let us now introduce a conditional probability of cleavage propagation, given initiation (P_{pr}). The hazard function is modified as follows:

$$H(K_J) = \frac{4K_J^{\;3}}{\hat{K}_o^{\;4}} P_{pr} \tag{22}$$

Wallin [8] has suggested the following functional form for the conditional probability of propagation:

$$P_{pr} = \begin{cases} \left(1 - \frac{K_{\min}}{K_J}\right)^3 & K_J > K_{\min} \\ 0 & K_J \le K_{\min} \end{cases} \tag{23}$$

Note that P_{pr} approaches unity when $K_J >> K_{\min}$.

Solving for cumulative probability gives,

$$\begin{aligned} F &= 1 - \exp\left[-\int_{K_{\min}}^{K} \frac{4K_J^{\ 3}}{\hat{K}_o^{\ 4}} \left(1 - \frac{K_{\min}}{K_J}\right)^3 dK_J \right] \\ &= 1 - \exp\left[-\left(\frac{K_J - K_{\min}}{\hat{K}_o}\right)^4 \right] \end{aligned} \tag{24}$$

which is a three-parameter Weibull distribution. This is consistent with the toughness distribution given by the Master Curve method [9]:

$$F = 1 - \exp\left[-\frac{B}{B_o}\left(\frac{K_{Jc} - K_{\min}}{K_o - K_{\min}}\right)^4 \right] \tag{25}$$

Where B is the crack front length and B_o is a reference length dimension (usually 25 mm). Equations (24) and (25) coincide if

$$\hat{K}_o = \left(K_o - K_{\min}\right)\left(\frac{B_o}{B}\right)^{1/4} \tag{26}$$

Under small-scale yielding conditions, the crack driving force is related to Weibull stress through Eq. (17). Thus the hazard function can be written as

$$H\left(\hat{\sigma}_w\right) = \frac{m\hat{\sigma}_w^{\ m-1}}{\hat{\sigma}_u^{\ m}} P_{pr} \tag{27}$$

Although Eq. (17) applies only in the case of small-scale yielding, the above expression is generally applicable because the Weibull stress is assumed to represent the true driving force for cleavage irrespective of whether or not K_J similitude applies.

Substituting Eqs. (23) and (27) into Eq. (20) results in the following expression for cumulative cleavage probability:

$$F = 1 - \exp\left[-\int_{K_{\min}}^{K_J} \frac{m\hat{\sigma}_w^{\,m-1}}{\hat{\sigma}_u^{\,m}} \frac{d\hat{\sigma}_w}{dK_J} \left(1 - \frac{K_{\min}}{K_J}\right)^3 dK_J \right] \tag{28}$$

The above expression assumes that the conditional probability of propagation is not influenced by constraint loss. This is a reasonable assumption, since the propagation term is most important as $K_J \rightarrow K_{\min}$, where K_J similitude will normally apply.

4 APPLICATION TO EXPERIMENTAL DATA

If the Weibull constants σ_u, $\sigma_{\min}$, and m are known for a given material, along with $K_{\min}$, Eq. (28) can be evaluated to compute cleavage probability for a given cracked body and applied load. Normally, an elastic-plastic finite element analysis of the configuration of interest is performed to compute stresses near the crack tip and the J-integral. The mesh near the crack tip must be sufficiently refined to capture the local stress field.

In order to apply this model to structural components, however, the material constants must first be calibrated with experimental data. This can be done by evaluating Eq. (28) iteratively to select the material constants that provide the best fit to experimental data. For a given material and temperature, two specimen configurations are normally used to calibrate the Weibull constants [1-4]. Once the model has been calibrated to two geometries, it can be used to predict the fracture behavior of other configurations.

Note that Eq. (28) contains a total of four material parameters: σ_u, $\sigma_{\min}$, m, and $K_{\min}$. Generally, it is neither practical nor appropriate to attempt to fit for all four constants to a given dataset. Instead, it is better to assign physically reasonable values to $\sigma_{\min}$ & $K_{\min}$, and then optimize σ_u and m to provide the best fit to the data. A suitable value for $\sigma_{\min}$ is typically in the range of 2 to 2.5 times the yield stress. The Master Curve method, as described in ASTM E 1921 [9], assumes a $K_{\min}$ value of 20 $\mathrm{MPa}\sqrt{\mathrm{m}}$, but higher values may be suitable, especially in the upper transition range.

Figure 1 shows experimental K_{Jc} data for SE(B) specimens with crack depth/width (a/W) ratios of 0.5 and 0.1 [5]. The curves represent fits of Eq. (28) for the two specimen configurations. Elastic-plastic finite element analyses were performed to provide the K_J and principal stress input into Eq. (28). In this case, a $K_{\min}$ value of 50 $\mathrm{MPa}\sqrt{\mathrm{m}}$ was assumed. As of this writing, further work is in progress to apply Eq. (28) to other materials and configurations.

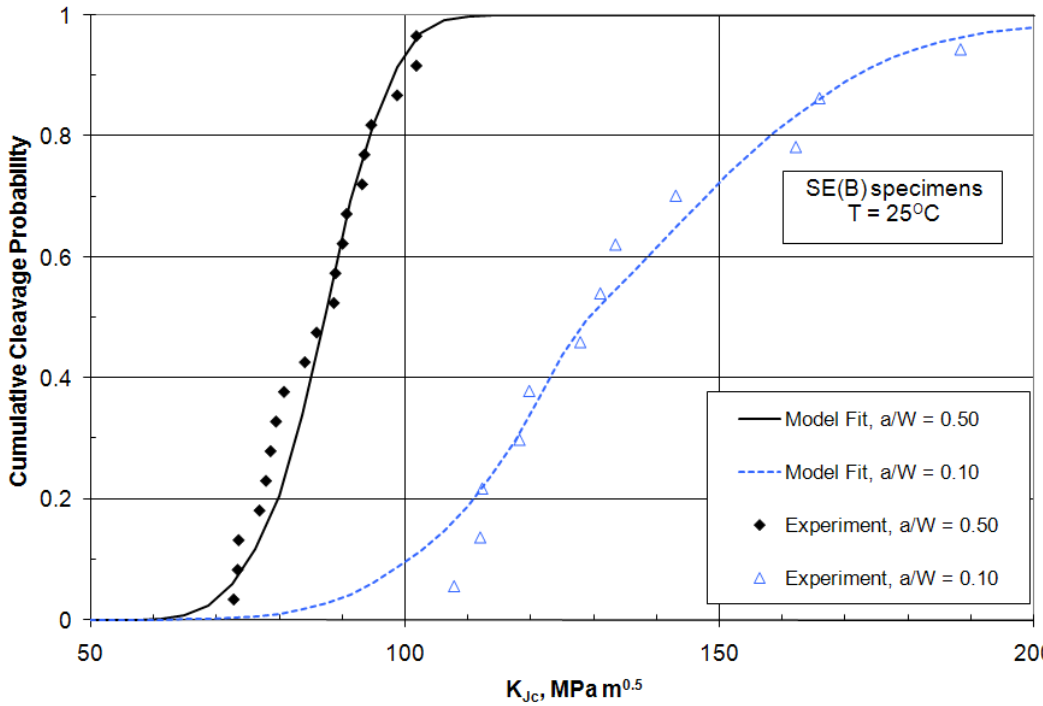

FIGURE 1. Experimental toughness data from Faleskog et al [5], which has been fit to Eq. (28) in the present study.

5 SUMMARY

In the present work, the Weibull stress model for cleavage was modified to incorporate a threshold toughness. A conditional probability of propagation, given cleavage initiation, was introduced into the hazard function. An extensive study is underway to apply this model to a variety of materials and configurations.

6 REFERENCES

[1] Gao. X., Dodds, R.H., Tregoning, R.L., Joyce, J.A., and Link, R.E., "A Weibull Stress Model to Predict Cleavage Fracture in Plates Containing Surface Cracks." *Fatigue and Fracture of Engineering Material Structures*, Vol. 22, 2004, pp. 481-493.

[2] Petti, J.P. and Dodds, R.H., "Coupling of the Weibull Stress Model and Macroscale Models to Predict Cleavage Fracture." *Engineering Fracture Mechanics*, Vol. 71, 2004, pp. 2079-2103.

[3] Petti, J.P. and Dodds, R.H., "Calibration of the Weibull Stress Scale Parameter, σ_u, using the Master Curve." *Engineering Fracture Mechanics*, Vol. 72, 2005, pp. 91-120.

[4] Bodgan, W., Petti, J.P. and Dodds, R.H., "Temperature Dependence of Weibull Stress Parameters using the Euro-Material." *Engineering Fracture Mechanics*, Vol. 73, 2006, pp. 1046-1069.

[5] Faleskog, J., Kroon, M., Ooberg, H. "A Probabilistic Model for Cleavage Fracture with a Length Scale—Parameter Estimation and Predictions of Stationary Crack Experiments." *Engineering Fracture Mechanics,* Vol. 71, 2004, pp. 57–79.

[6] Bordet, S.R., Karstensen, A.D., Knowles, D.M., and Weisner, C.S., "A New Statistical Local Criterion for Cleavage Fracture in Steel, Part I: Model Presentation." *Engineering Fracture Mechanics*, Vol. 72, 2005, pp. 435-452.

[7] Bordet, S.R., Karstensen, A.D., Knowles, D.M., and Weisner, C.S., "A New Statistical Local Criterion for Cleavage Fracture in Steel, Part I: Model Presentation." *Engineering Fracture Mechanics*, Vol. 72, 2005, pp. 453-474.

[8] Wallin, K. "Microscopic Nature of Brittle Fracture." *Journal de Physique*, Vol. 3, 1993, pp. 575-584.

[9] E 1921-05, "Standard Test Method for Determination of Reference Temperature, T_o, for Ferritic Steels in the Transition Range." ASTM International, Philadelphia, PA, 2005.