About Accumulation Rules for Life Time Prediction under Variable Loading

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Abstract

A general functional form of temporal strength conditions under variable loading is employed to formulate several new accumulation rules. Unlike the well known Robinson rule of linear accumulation of partial life-times, the new rules are sensible to the load order. Comparison with some experiments show that they much better fit experimental results.

1 Introduction to accumulation rules

Let $t = t^{*0}(\sigma)$ be a material durability (life-time) diagram under a constant stress state $\sigma = \sigma_{ij} = const$. If the stress state is not constant but a function of time (process) $\sigma = \sigma_{ij}(\tau)$, then the Robinson model of linear accumulation of partial life-times, [1, 2] (see also [3, 4]), gives equation

$$\int_0^{t^*} \frac{d\tau}{t^{*0}(\sigma(\tau))} = 1. \tag{1}$$

for life-time $t^* = t^*(\sigma)$ under the process $\sigma_{ij}(\tau)$.

The well-known non-sensitivity of the Robinson model to the order of load application can be readily observed from (1): applying first higher and then lower load or wise-versa lead to the same life-time. However many experiments on variable loading show that this is generally not the case. Although some modifications of the Robinson model to address this issue have been reported in the literature, no one seems to be widely accepted.

To propose a new accumulation rule sensible to the load order, we first remark that, as shown in [5], any model for time-dependent strength and life-time analysis can be expressed in the form

$$\underline{\Lambda}^T(\sigma; t) = 1.$$

Here $\underline{\Lambda}^T(\sigma; t)$ is the temporal *normalised equivalent stress*, NES, that for a given process $\sigma_{ij}(\tau)$ and an instant t is defined as *infimum of numbers* $\Lambda' > 0$

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such that there is no rupture at or before the time t under the process $\frac{1}{\Lambda''}\sigma_{ij}(\tau)$ for any $\Lambda'' > \Lambda'$; if there is no such Λ', we take $\underline{\Lambda}^T(\sigma;t) = \infty$.

So defined, the temporal *normalised equivalent stress functional*, NESF, $\underline{\Lambda}^T$ is a material characteristics and should be identified from experiments and/or a model. The definition implies that the functional $\underline{\Lambda}^T(\sigma;t)$ is non-negative positively-homogeneous of the order +1 in the first argument and non-decreasing in the second argument, that is

$$\underline{\Lambda}^T(k\sigma;t) = k\underline{\Lambda}(\sigma;t) \geq 0 \quad \text{for any } k > 0, \quad \underline{\Lambda}^T(\sigma;t_2) \geq \underline{\Lambda}^T(\sigma;t_1) \text{ if } t_2 > t_1.$$

These properties make $\underline{\Lambda}$ uniquely determinable and narrow down the admissible forms of $\underline{\Lambda}$.

Particularly, the Robinson rule (1) can also be re-written in the form

$$\underline{\Lambda}^{TR}(\sigma;t) = 1, \tag{2}$$

where $\underline{\Lambda}^{TR}(\sigma;t)$ is the minimal solution Λ of the equation

$$\int_0^{t^*} \frac{d\tau}{t^{*0}(\sigma(\tau)/\Lambda)} = 1.$$

In Section 2 we give an explicit solution of this equation leading to an explicit expression for $\underline{\Lambda}^{TR}$ obtained in [5] for the Basquin-type durability diagram.

On the other hand, the temporal rupture criterion under constant uniaxial loading $\sigma = \mathit{const}$ starting at $t = 0$ can be written as $|\sigma| = \sigma^*(t)$, where the function $\sigma^*(t)$ gives another form of the durability diagram and is inverse to the function $t^{*0}(\sigma)$, i.e. the equality $|\sigma| = \sigma^*(t^{*0}(\sigma))$ is identically satisfied for any σ. Similarly, the temporal rupture criterion under constant multiaxial lading $\sigma_{ij} = \mathit{const}$ started at $t = 0$, can be written as $|\sigma| = \sigma^*(\tilde{\sigma};t)$, which can be reformulated in terms of the NES as $\underline{\Lambda}^T(\sigma;t) = 1$, where

$$\underline{\Lambda}^T(\sigma;t) := \frac{|\sigma|}{\sigma^*(\tilde{\sigma};t)}. \tag{3}$$

Here $|\sigma|$ is a matrix norm of the tensor σ_{ij}, e.g., $|\sigma| = \sqrt{\sum_{i,j=1}^3 \sigma_{ij}\sigma_{ij}}$, and $\tilde{\sigma}_{ij} := \sigma_{ij}/|\sigma|$ is the unit tensor presenting the stress tensor σ_{ij} shape. Making formal manipulations, we have from (3),

$$\begin{aligned}\underline{\Lambda}^T(\sigma;t) &= \frac{|\sigma|}{\sigma^*(\tilde{\sigma};0)} + \int_0^t |\sigma| \frac{\partial}{\partial\xi}\left[\frac{1}{\sigma^*(\tilde{\sigma};\xi)}\right] d\xi \\ &= \frac{|\sigma|}{\sigma^*(\tilde{\sigma};0)} + \int_0^t |\sigma| \frac{\partial}{\partial t}\left[\frac{1}{\sigma^*(\tilde{\sigma};t-\tau)}\right] d\tau = \frac{\partial}{\partial t}\int_0^t \frac{|\sigma|}{\sigma^*(\tilde{\sigma};t-\tau)} d\tau. \end{aligned} \tag{4}$$

Hinted by (4), we extend the form for $\underline{\Lambda}^T$ also to variable processes $\sigma_{ij}(\tau)$, such that $\sigma_{ij}(\tau) = 0$ at $\tau < 0$, introducing rupture criterion

$$\underline{\Lambda}_1^{TS}(\sigma;t) = 1,$$

with the following NESF

$$\underline{\Lambda}_1^{TS}(\sigma;t) := \max_{t'\le t} \hat{\underline{\Lambda}}_1^{TS}(\sigma;t'), \quad \hat{\underline{\Lambda}}_1^{TS}(\sigma;t') := \frac{\partial}{\partial t'}\int_{-0}^{t'} \frac{|\sigma(\tau)|}{\sigma^*(\tilde{\sigma}(\tau);t'-\tau)}d\tau,$$

that can be interpreted as a linear accumulation rule for partial NES, a counterpart of the Robinson rule of linear accumulation of partial life-times. If the durability diagram can be we presented as a product,

$$\sigma^*(\tilde{\sigma};t) = \sigma^0(\tilde{\sigma})\sigma^{**}(t), \tag{5}$$

where the scalar functions σ^0 and σ^{**} are material characteristics, then $\hat{\underline{\Lambda}}_1^{TS}$ after integrating by parts is reduced to

$$\begin{aligned}
\hat{\underline{\Lambda}}_1^{TS}(\sigma;t') &= \frac{|\sigma(t')|}{\sigma^0(\tilde{\sigma}(t'))\sigma^{**}(0)} + \int_{-0}^{t'} \frac{|\sigma(\tau)|}{\sigma^0(\tilde{\sigma}(\tau))}\frac{\partial}{\partial t'}\left[\frac{1}{\sigma^{**}(t'-\tau)}\right]d\tau \\
&= \frac{|\sigma(t')|}{\sigma^0(\tilde{\sigma}(t'))\sigma^{**}(0)} - \int_{-0}^{t'} \frac{|\sigma(\tau)|}{\sigma^0(\tilde{\sigma}(\tau))}\frac{\partial}{\partial \tau}\left[\frac{1}{\sigma^{**}(t'-\tau)}\right]d\tau \\
&= \int_{-0}^{t'} \frac{1}{\sigma^{**}(t'-\tau)} d\frac{|\sigma(\tau)|}{\sigma^0(\tilde{\sigma}(\tau))} \ .
\end{aligned}$$

Here the last integral should be understood in the Stiltjes sense if the function $\sigma(t)$ is discontinuous.

By similar argument one can also arrive at the following more general non-linear (power-type) rule of NES accumulation,

$$\underline{\Lambda}_\beta^{TS}(\sigma;t) := \max_{t'\le t} \hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t') = 1, \tag{6}$$

$$\begin{aligned}
\hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t') &:= \left[\frac{\partial}{\partial t'}\int_{-0}^{t'} \left|\frac{\sigma(\tau)}{\sigma^*(t'-\tau)}\right|^\beta d\tau\right]^{1/\beta} \\
&= \left[\int_{-0}^{t'} \frac{1}{[\sigma^{**}(t'-\tau)]^\beta} d\left|\frac{\sigma(\tau)}{\sigma^0(\tilde{\sigma}(\tau))}\right|^\beta\right]^{1/\beta} ,
\end{aligned} \tag{7}$$

where $\beta \neq 0$ is a material constant and the last equality holds under condition (5). For a constant multiaxial lading $\sigma_{ij} = const$ started at $t = 0$, expresion (7) reduces to (3) for any β.

One can see that a linear combination of the terms (7) leads to even more general non-linear rule of NES accumulation

$$\underline{\Lambda}_{comb}^{TS}(\sigma;t) := \max_{t'\le t} \hat{\underline{\Lambda}}_{comb}^{TS}(\sigma;t') = 1, \tag{8}$$

$$\hat{\underline{\Lambda}}_{comb}^{TS}(\sigma;t') := \sum_{n=1}^{N} \alpha_n \hat{\underline{\Lambda}}_{\beta_n}^{TS}(\sigma;t'), \tag{9}$$

where the numbers β_n, α_n are material parameters such that $\sum_n^N \alpha_n = 1$, to ensure that $\underline{\Lambda}_{comb}^{TS}(\sigma;t)$ reduces to (3) for $\sigma_{ij} = const$ started at $t = 0$.

Note that for fatigue the counterparts of the accumulation rules presented in this section were given in [6].

2 Accumulation rules for Basquin-type durability diagram

2.1 General loading

Consider the case of the durability diagram under constant uniaxial or multiaxial loading $\sigma = const$ starting at $t > 0$ described by the Basquin-type relation

$$|\sigma| = \sigma^*(\tilde{\sigma};t) \text{ where } \sigma^*(t) = \sigma^0(\tilde{\sigma})t^{-1/b}, \tag{10}$$

Here b is a material constant and $\sigma^0 = \sigma^0(\tilde{\sigma})$ is a material characteristics possibly depending on the unit tensor $\tilde{\sigma}$. Note that (10) is a special case of relation (5) with $\sigma^{**}(t) = t^{-1/b}$. Durability diagram (10) can be also presented as

$$t^{*0}(\sigma) = \left(\frac{|\sigma|}{\sigma^0}\right)^{-b}. \tag{11}$$

Then the Robinson rule (1) can be written in form (2) of the NES has the following form [5],

$$\underline{\Lambda}^{TR}(\sigma;t) := \left[\int_0^t \left|\frac{\sigma(\tau)}{\sigma^0(\tilde{\sigma}(\tau))}\right|^b d\tau\right]^{1/b} = \left[\int_0^t \frac{d\tau}{t^{*0}(\sigma(\tau))}\right]^{1/b}.$$

For a uniaxial process σ^0 is a constant if $\sigma(\tau)$ does not change sign.

On the other hand, the functional $\hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t')$ for durability diagram (10) becomes

$$\begin{aligned} \hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t') &= \left[\frac{\partial}{\partial t'}\int_{-0}^{t'} \left|\frac{\sigma(\tau)}{\sigma^0(\tilde{\sigma}(\tau))}\right|^\beta (t'-\tau)^{\beta/b} d\tau\right]^{1/\beta} \\ &= \left[\int_{-0}^{t'} (t'-\tau)^{\beta/b} d\left|\frac{\sigma(\tau)}{\sigma^0(\tilde{\sigma}(\tau))}\right|^\beta\right]^{1/\beta}. \end{aligned} \tag{12}$$

Particularly, for $\beta = b$ expression (12) implies that $\hat{\underline{\Lambda}}_b^{TS}(\sigma;t) = \underline{\Lambda}^{TR}(\sigma;t)$, i.e., for the Basquin durability diagram, the Robinson linear summation rule for partial life-times is a special case of the non-linear rule of NES accumulation (6)-(7) with $\beta = b$.

2.2 Uniaxial step-wise loading

Consider a uniaxial loading $\sigma(\tau) = \begin{cases} 0, & \tau < t_0 = 0 \\ \sigma_k, & t_{k-1} \le \tau < t_k, \quad 0 < k < K-1, \\ \sigma_K, & t_{K-1} \le \tau \end{cases}$

$\sigma(\tau) \ge 0$. Under such process σ^0 is constant in the Basquin durability diagram (11), and the Robinson rule (1) gives the following equation for cumulative durability $t = t_K$,

$$\sum_{k=1}^{K} r_k = 1,$$

where

$$r_k := \frac{t_k - t_{k-1}}{t^{*0}} = (t_k - t_{k-1}) \left(\frac{\sigma_k}{\sigma^0}\right)^b.$$

In terms of the NESF $\underline{\Lambda}^{TR}$, this is equivalent to equation (2), where

$$\underline{\Lambda}^{TR}(\sigma;t) = \underline{\Lambda}_b^{TS}(\sigma;t) = \hat{\underline{\Lambda}}_b^{TS}(\sigma;t)$$
$$= \frac{1}{\sigma^0}\left[\sum_{k=1}^{k'-1} \sigma_k^b(t_k - t_{k-1}) + \sigma_{k'}^b(t - t_{k'-1})\right]^{1/b} = \left[\sum_{k=1}^{k'-1} r_k + r_{k'}\right]^{1/b}, \quad (13)$$

k' is such that $t_{k'-1} < t \le t_{k'}$, $r_{k'} := (t - t_{k'-1})\left(\sigma_{k'}/\sigma^0\right)^b$.

On the other hand, for the same process the general power-type accumulation NES is

$$\underline{\Lambda}_\beta^{TS}(\sigma;t) = \max_{0\le t'\le t} \hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t'),$$
$$\hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t') = \frac{1}{\sigma^0}\left[\sum_{t_{k-1}<t'} (\sigma_k^\beta - \sigma_{k-1}^\beta)(t' - t_{k-1})^{\beta/b}\right]^{1/\beta}. \quad (14)$$

Since $\underline{\Lambda}_b^{TS}(\sigma;t) = \underline{\Lambda}^{TR}(\sigma;t)$, it is not sensitive to the load order, while $\underline{\Lambda}_\beta^{TS}(\sigma;t)$ is, if $\beta \ne b$. To illustrate this, let us consider the two-step loading, $K = 2$, and write (14) for $t = t_2 > t_1$ as

$$\hat{\underline{\Lambda}}_\beta^{TS}(\sigma;t_2) = \frac{1}{\sigma^0} \max_{t_1\le t'\le t_2} \left[\sigma_1^\beta t'^{\beta/b} + (\sigma_2^\beta - \sigma_1^\beta)(t' - t_1)^{\beta/b}\right]^{1/\beta}$$
$$= \max_{0\le r'\le r_2} \left[(r_1 + s^b r')^{\beta/b} + (1 - s^\beta) r'^{\beta/b}\right]^{1/\beta}. \quad (15)$$

where $s = \sigma_1/\sigma_2$ is the parameter of the load order, i.e. $s < 1$ corresponds to the low-to-high order of loading, while $s > 1$ to the high-to-low order. Particularly,

$$\underline{\Lambda}_1^{TS}(\sigma;t_2) = \frac{1}{\sigma^0} \max_{t_1\le t'\le t_2} \left[\sigma_1 t'^{1/b} + (\sigma_2 - \sigma_1)(t' - t_1)^{1/b}\right]$$
$$= \max_{0\le r'\le r_2} \left[(r_1 + s^b r')^{1/b} + (1 - s) r'^{1/b}\right].$$

By (13),

$$\underline{\Lambda}_b^{TS}(\sigma;t_2) = \frac{1}{\sigma^0}\left[\sigma_1^b t_1 + \sigma_2^b(t_2 - t_1)\right]^{1/b} = [r_1 + r_2]^{1/b} \tag{16}$$

Equating (15) and (16) to 1, we obtain the relation between the partial life times r_1 and r_2 in the linear Robinson and NES accumulation rules. In Fig. 1 the straight line corresponds to the Robinson rule, and the curves correspond to the linear NES accumulation rule for $b = 5$ and labelled with the value of s. The high dependence on s shows the high dependence of the durability predictions on the order of loading.

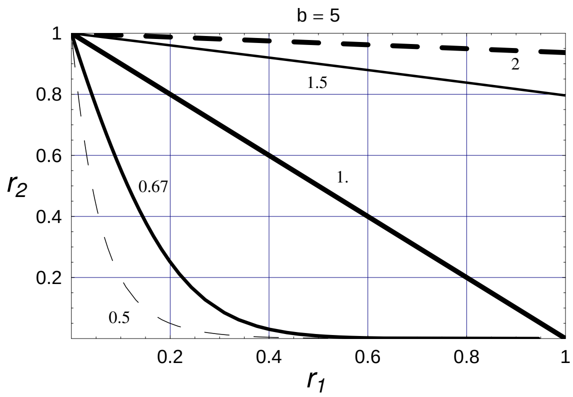

Figure 1: Deviation from the Robinson rule (straight line) associated with NESF $\underline{\Lambda}_1^{TS}$ for $b = 5$ at several values of s .

3 Comparison with experiment

Let us compare some experimental results with the durability predictions given by the Robinson linear accumulation and by the NES linear accumulation rules. Some durability experiments for an aluminium alloy at 180°C under uniaxial constant and variable (step-wise) stress processes are reported in [7]. Fitting the results from [7, Table 2] for constant loading to the Basquin durability diagram (11), we obtained the following values for its parameters, $\sigma^0 = 56109\frac{lb}{in^2}h^{1/b}$, $b = 5.68$.

Fig. 2-7 show the graphs of $[\hat{\underline{\Lambda}}_1^{TS}(\sigma;t)]^b$, $[\underline{\Lambda}_1^{TS}(\sigma;t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma;t)]^b$ vs. time, calculated for the 2-step Low-to-High and High-to-Low tests from [7, Tables 3, 4], as well as the test rupture times. The loading program is given in the figure captions. The graph for $[\hat{\underline{\Lambda}}_1^{TS}(\sigma;t)]^b$ and $[\underline{\Lambda}_1^{TS}(\sigma;t)]^b$ coincide except on small parts for $s > 1$.

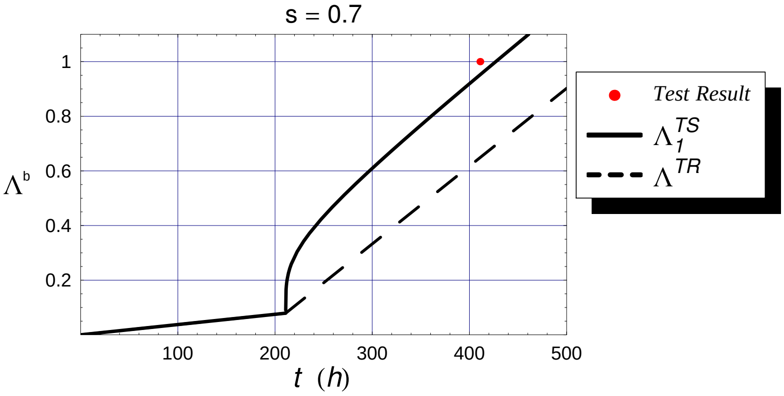

Figure 2: $[\underline{\Lambda}_1^{TS}(\sigma;t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma;t)]^b$ vs. t;
[211h @ 14000 lb/in^2+200h @ 20000 lb/in^2]

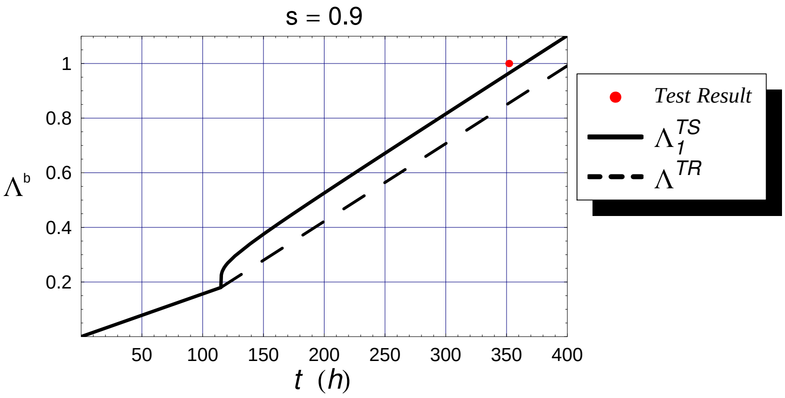

Figure 3: $[\underline{\Lambda}_1^{TS}(\sigma;t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma;t)]^b$ vs. t;
[115h @ 18000 lb/in^2+237h @ 20000 lb/in^2]

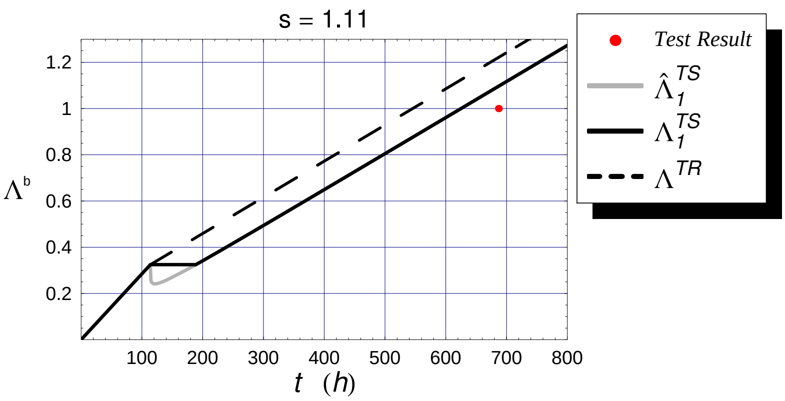

Figure 4: $[\underline{\Lambda}_1^{TS}(\sigma;t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma;t)]^b$ vs. t;
[114h @ 20000 lb/in^2+573h @ 18000 lb/in^2]

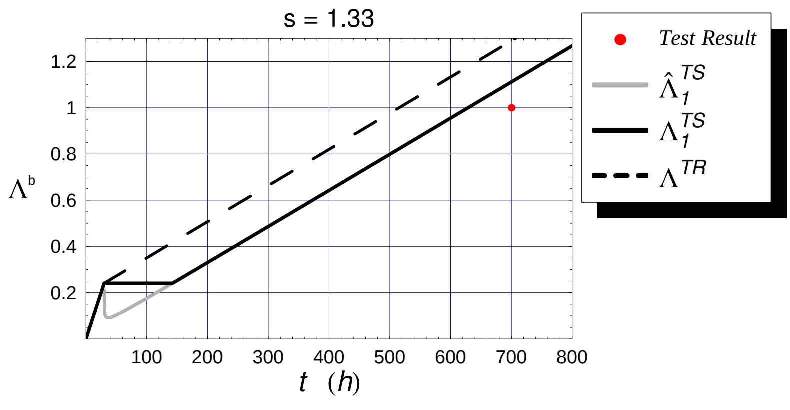

Figure 5: $[\underline{\Lambda}_1^{TS}(\sigma; t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma; t)]^b$ vs. t;
[30h @ 24000 lb/in^2+670h @ 18000 lb/in^2]

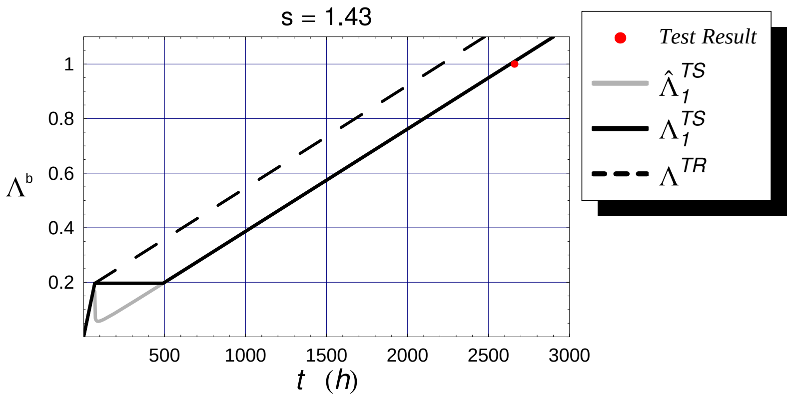

Figure 6: $[\underline{\Lambda}_1^{TS}(\sigma; t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma; t)]^b$ vs. t;
[69h @ 20000 lb/in^2+2590h @ 14000 lb/in^2]

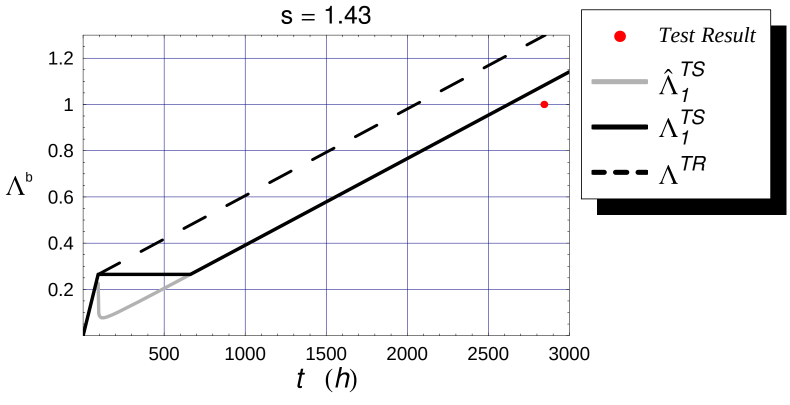

Figure 7: $[\underline{\Lambda}_1^{TS}(\sigma; t)]^b$ and $[\underline{\Lambda}^{TR}(\sigma; t)]^b$ vs. t;
[93h @ 20000 lb/in^2+2751h @ 14000 lb/in^2]

Conclusion

One can see that discrepancy between the life-time theoretical predictions and considered experiments is from 2 to 19 times less, when using linear NES accumulation, than when the Robinson linear rule of partial life-times accumulation is used. Thus the linear NES accumulation rule seems to be a viable alternative to the Robinson rule. Further improvements in the life time prediction can be obtained using e.g. the combined NES accumulation rule (7)-(9).

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