Interaction of a Griffith Crack in a Three-Phase Circular Inclusion with two Imperfect Interfaces

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Abstract: In this work, a semi-analytical method is presented for the problem of a pre-existing radial Griffith crack embedded within the interphase layer surrounding a circular inclusion. Novel to this work is that the bonding at the inclusion-interphase interface and the interphase-matrix interface is considered to be imperfect with the assumption that the interface imperfections are constant. Employing complex variable techniques, we derive series representations for the corresponding stress functions inside the inclusion, in the interphase layer and the surrounding matrix. The advantage of the series method over other methods, such as the dislocation density method, is that in the former case the resulting expressions are linear and can be solved readily whereas in the latter case the method leads to cumbersome integral equations which are often numerically difficult to solve. Stress intensity factor (SIF) calculations are performed at the crack tips for different material property combinations, imperfect interface conditions and crack locations under mode I loading. The results not only provide for a quantitative description of the interaction between a radial interphase crack and a three-phase inclusion with imperfect interfaces but the results clearly demonstrate the significance of how two imperfect boundaries can influence crack behaviour.

1.0 Introduction

With the rapid development of sophisticated multi-phase materials the interaction between material defects and so-called coated inclusions are becoming increasingly important in the manufacture of composite materials. The performance of these advanced materials depends not only on the properties of the constituent phases but also on the quality of adhesion that exists between matrix and fiber. In fact, when the matrix bonds to the fiber surface a thin intermediate zone commonly known as the interphase layer is created. Alternatively, this interphase layer can be introduced as a separate material during the design stage so as to tailor the performance of the advanced material to any particular condition. In any case, the interphase layer can be defined as a non-uniform, anisotropic region of finite thickness consisting of structural defects (such as microcracks, voids and other impurities) and having distinct mechanical properties as compared to fiber or matrix.

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In the last several years a tremendous amount of work has been devoted to understanding the mechanical behaviour surrounding the interaction problem between material defects and three-phase inclusion systems (see, for example, [1-4]). However, the limitations of the aforementioned works are primarily based on the assumption of perfect bonding between all constituent boundaries. In many practical problems various kinds of interface damage (such as interface debonding) makes the perfect bonding model inadequate. In these kinds of situations, it becomes necessary to model the interface as an *imperfectly bonded interface.*

Recently, Kim and Sudak [5] addressed the problem of a three-phase circular inclusion with an imperfect interface interacting with a radial matrix crack. They showed that having an imperfect inclusion/interphase interface significantly affects the stress intensity factors of the radial matrix crack. While these results, with only one imperfect interface, are important in the design of advanced materials, it is conceivable and practically relevant that composite materials, in particular, fiber-reinforced materials could contain two or more imperfect interfaces (see, for example, [6]).

In the present work, a general semi-analytical approach is developed which, for the first time, takes into account simultaneously the effects of two imperfect interface conditions and their subsequent effect on a pre-existing crack located in the interphase layer. The results clearly demonstrate the significance of the interphase layer as well the influence of imperfect material interfaces on crack behaviour for a variety of material property combinations, crack positions and crack size. It should be noted that the advantage of the current series method as compared to the dislocation density method used model crack problems is that in the former case the series method is simple and easy to use whereas in the latter case it is often cumbersome and numerically challenging to implement.

2.0 Formulation

Consider a domain in R^2, infinite in extent, containing a single circular inclusion which is bonded to a matrix through a single coaxial circular interphase layer. The linearly elastic materials occupying the inclusion, the interphase layer and the matrix are assumed to be homogeneous and isotropic with associated shear moduli $\mu_0(>0)$, $\mu_1(>0)$ and $\mu_2(>0)$, respectively. The inclusion, with center at the origin of the coordinate system and radius R_0, occupies a domain denoted by S_0. The interphase layer, with radius R_1 is denoted by S_1 and contains a pre-existing radial Griffith crack of length $2l$. The surrounding matrix is denoted by S_2. The inclusion-interphase interface and the interphase-matrix interface are denoted by the curves $\Gamma_k(k=0,1)$, respectively (see Figure 1). Furthermore, unless otherwise stated, the subscripts 0,1,2 will be used to denote quantities in S_0, S_1 and S_2, respectively.

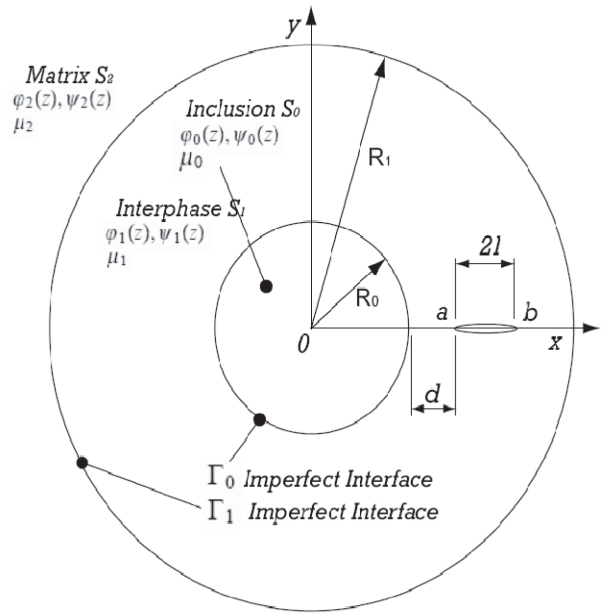

Figure1. Geometry of the problem

Let (x,y) denote a generic point in R^2 and $z=x+iy=re^{i\theta}$ the complex coordinate. Then for plane deformation, the elastic stresses and associated displacements can be given in terms of two potential functions $\phi(z)$ and $\psi(z)$ as [7]

$$\begin{aligned} 2\mu\left(u_r + iu_\theta\right) &= e^{-i\theta}\left[\kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}\right], \\ \sigma_{rr} + \sigma_{\theta\theta} &= 2\left[\varphi'(z) + \overline{\varphi'(z)}\right], \\ \sigma_{rr} - i\sigma_{r\theta} &= \varphi'(z) + \overline{\varphi'(z)} - e^{2i\theta}\left[\overline{z}\varphi''(z) + \psi'(z)\right], \end{aligned} \tag{1}$$

where $\kappa = (3\text{-}4v)$ for plane strain and $\kappa=(3\text{-}v)/(1+v)$ for plane stress. Here μ and v are the shear modulus and Poisson's ratio, respectively. In addition, the resultant force acting on an arbitrary arc AB in the elastic body is given by [7]

$$F_x + iF_y = -i\left[\varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}\right]_A^B \quad , \tag{2}$$

where $[f(*)]_A^B = f(B) - f(A)$ is independent of the path.
In order to describe the variability in bonding at the inclusion-interphase layer interface and the interphase layer-matrix interface one of the more widely accepted mechanical descriptions is based on the premise that tractions are continuous but displacements are discontinuous across the interface (see [8]). In this context, the displacement jumps are assumed to be proportional to their respective traction components. In view of this, the corresponding boundary value problem along Γ_0 and Γ_1 can be formulated as follows

$$\|\sigma_{rr} - i\sigma_{r\theta}\| = 0,$$
$$\sigma_{rr} = m_0 \|u_r\| \quad , \quad \sigma_{r\theta} = n_0 \|u_\theta\| \quad \Gamma_0,$$
$$\|\sigma_{rr} - i\sigma_{r\theta}\| = 0, \tag{3}$$
$$\sigma_{rr} - m_1 \|u_r\| \quad , \quad \sigma_{r\theta} - n_1 \|u_\theta\| \quad \Gamma_1,$$

where m and n are non-negative and constant interface parameters. Physically, these parameters characterize the strength, stiffness and overall degree of adhesion along the material interface and they are described by a simple constitutive relationship (for details see [9]). The expressions $\|*\|=\|*\|_1 - \|*\|_0$ and $\|*\|=\|*\|_2 - \|*\|_1$ denote the jump across Γ_k,(k=0,1). In addition, we note that if $m_0=n_0\to 0$ and $m_1=n_1\to 0$ (Eq.3) represent the traction free boundary condition and if $m_0=n_0\to\infty$ and $m_1=n_1\to\infty$ (Eq.3) corresponds to perfectly bonded interfaces. Hence, in view of (Eqs.1-2), the boundary value problem can be expressed in terms of six complex analytic functions $\phi_i(z)$ and $\psi_i(z)$ (i=0,1,2) as follows

$$\varphi_1'(z)+\overline{\varphi_1'}\left(\frac{R_0^2}{z}\right)-z\varphi_1''(z)-\frac{z^2}{R_0^2}\psi_1'(z)=\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z),\quad z\in\Gamma_0,$$

$$\begin{aligned}
&(m_0-n_0)(\kappa_1+1)\frac{R_0}{z}\varphi_1(z)+(m_0+n_0)(\kappa_1+1)\frac{z}{R_0}\overline{\varphi_1}\left(\frac{R_0^2}{z}\right)\\
&=4\mu_1\left[\varphi_0'(z)+\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)-z\varphi_0''(z)-\frac{z^2}{R_0^2}\psi_0'(z)\right]+(m_0-n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{R_0}{z}\varphi_0(z)\\
&+(m_0-n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\overline{\varphi_0'}\left(\frac{R_0^2}{z}\right)+\frac{R_0}{z}\overline{\psi_0}\left(\frac{R_0^2}{z}\right)\right]\\
&+(m_0+n_0)(1+\kappa_0\frac{\mu_1}{\mu_0})\frac{z}{R_0}\overline{\varphi_0}(\frac{R_0^2}{z})+(m_0+n_0)(1-\frac{\mu_1}{\mu_0})\left[R_0\varphi_0'(z)+\frac{z}{R_0}\psi_0(z)\right]\\
&+4m_0R_0\mu_1\varepsilon_1+\frac{2\mu_1(m_0+n_0)(\varepsilon_2-i\varepsilon_3)}{R_0}z^2+\frac{2\mu_1(m_0-n_0)(\varepsilon_2+i\varepsilon_3)R_0^3}{z^2},\quad z\in\Gamma_0,
\end{aligned}\tag{4}$$

$$\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)=\varphi_1'(z)+\overline{\varphi_1'}\left(\frac{R_1^2}{z}\right)-z\varphi_1''(z)-\frac{z^2}{R_1^2}\psi_1'(z),\quad z\in\Gamma_1,$$

$$\begin{aligned}
&(m_1-n_1)(\kappa_1+1)\frac{R_1}{z}\varphi_1(z)+(m_1+n_1)\frac{z}{R_1}(\kappa_1+1)\overline{\varphi_1}\left(\frac{R_1^2}{z}\right)\\
&=-4\mu_1\left[\varphi_2'(z)+\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)-z\varphi_2''(z)-\frac{z^2}{R_1^2}\psi_2'(z)\right]\\
&+(m_1-n_1)\left(1-\frac{\mu_1}{\mu_2}\right)\left[R_1\overline{\varphi_2'}\left(\frac{R_1^2}{z}\right)+\frac{R_1}{z}\overline{\psi_2}\left(\frac{R_1^2}{z}\right)\right]+(m_1-n_1)\left(1+\frac{\mu_1}{\mu_2}\kappa_2\right)\varphi_2(z)\\
&+(m_1+n_1)\frac{z}{R_1}\left(1+\frac{\mu_1}{\mu_2}\kappa_2\right)\overline{\varphi_2}\left(\frac{R_1^2}{z}\right)+(m_1+n_1)\frac{z}{R_1}\left(1-\frac{\mu_1}{\mu_2}\right)\left[\psi_2(z)+\frac{R_1}{z}\varphi_2'(z)\right],\quad z\in\Gamma_1.
\end{aligned}$$

In addition to the boundary conditions given by (Eq.3), the traction free condition at the crack face via (Eq.2) is given by

$$\varphi_1(z)^+ + z\overline{\varphi_1'(z)^+} + \overline{\psi_1(z)^+} = 0, \qquad z \in 2l^+,$$
$$\varphi_1(z)^- + z\overline{\varphi_1'(z)^-} + \overline{\psi_1(z)^-} = 0, \qquad z \in 2l^-. \tag{5}$$

where the '+' and '-' refers to the upper and lower crack face respectively. Note that all functions appearing on both sides of (Eqs. 4-5) have been written with the understanding that they may be represented as power series expansions in the variable z. Thus, (Eqs.4-5) require that the series expansions on both sides have the same coefficients. Furthermore, the remote loadings are defined as

$$\varphi_2(z) = Az + O(1), \qquad \psi_2(z) = Bz + O(1) \quad , |z| \to \infty, \tag{6}$$

where A is a given real number and B is a given complex number determined by the uniform remote principal stresses

$$\sigma_{xx}^{\infty} + \sigma_{yy}^{\infty} = 4A, \qquad \sigma_{xx}^{\infty} - \imath\sigma_{xy}^{\infty} = 2A - B. \tag{7}$$

2.2 SERIES REPRESENTATION

A convenient method to analyze problems with circular boundaries is the series method. Let us now consider the functions $\phi_0(z)$ and $\psi_0(z)$ defined in S_0. They are analytic within the inclusion and as such can be expanded in a standard Taylor series in the domain S_0 as follows

$$\varphi_0(z) = \sum_{k=0}^{\infty} a_k z^k, \qquad \psi_0(z) = \sum_{k=0}^{\infty} b_k z^k \quad , |z| < R_0. \tag{8}$$

The next pair of complex functions those being $\phi_2(z)$ and $\psi_2(z)$ are analytic in S_2 and can subsequently be expressed as Laurent series in the domain S_2 as follows

$$\varphi_2(z) = Az + \sum_{k=1}^{\infty} c_k z^{-k}, \qquad \psi_2(z) = Bz + \sum_{k=1}^{\infty} d_k z^{-k} \quad , |z| > R_1. \tag{9}$$

The remaining set of complex functions, namely $\phi_1(z)$ and $\psi_1(z)$ are not analytic in the domain S_1 due to the presence of the crack. Thus, the conventional Laurent series is not applicable. To overcome this difficulty, the technique of analytic continuation is employed so that these potentials can be expressed in terms of two new complex functions which are analytic inside S_1 and can subsequently be expanded with standard series methods. Let Ω denote the domain S_1 minus the crack; in other words, Ω represents the annular region between Γ_0 and Γ_1 minus the crack. So then from (Eqn. 5) we have that

$$\varphi_1^+(z) - [z\overline{\varphi_1'(z)} + \overline{\psi_1(z)}]^+ = \varphi_1^-(z) - [z\overline{\varphi_1'(z)} + \overline{\psi_1(z)}]^-, \qquad z\epsilon 2l \tag{10}$$

Thus, as we approach the crack from either the upper or lower half plane, the above equality is satisfied which implies the function is continuous across the crack. Hence, let us define a new function, say $X(z)$, in S_1 as

$$\psi_1(z) = -\overline{X}(z) + \overline{\varphi_1}(z) - z\varphi_1'(z). \tag{11}$$

Then it is clear from (Eqn. 11) that $X(z)$ is continuous across the crack and regular in Ω. Consequently, it can be expanded into a Laurent series in Ω
In addition, if we add the expressions in (Eqn. 5) we obtain

$$[\varphi_1\ (z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)}]^+ + [\varphi_1\ (z) + z\overline{\varphi_1'(z)} + \overline{\psi_1(z)}]^- = 0, z\epsilon 2l. \tag{12}$$

Noting that the each term on the left hand side is always finite then using Plemelj's expressions [7] we can define a second function, say $Y(z)$, such that

$$\varphi_1(z) = \frac{1}{2\sqrt{(z-a)(z-b)}}Y(z) + \frac{1}{2}X(z), \tag{13}$$

Thus, $X(z)$ and $Y(z)$ are two new analytic functions which are continuous across the crack and analytic in Ω. Consequently, they can be expanded into a Laurent series in Ω as

$$X(z) = \sum_{k=-\infty}^{\infty} e_k z^k, \quad z \in \Omega,$$

$$Y(z) = \sum_{k=-\infty}^{\infty} f_k z^k, \quad z \in \Omega, \tag{14}$$

Note that the term F(z)=1/(2√(z-a)(z-b)) in (Eq. 13) is a multi-valued function across the crack but analytic in the domain $\Gamma_0 \cup \Gamma_1 \cup S_1$. Consequently, it can be expanded on the interfaces in a conventional series. Thus, F(z) can be defined as follows

$$F(z) = \begin{cases} \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2}(z-a)^{-\frac{1}{2}}(z-b)^{-\frac{1}{2}} = \sum_{k=0}^{\infty} g_k z^k, & |z| < a, \\ \frac{1}{2\sqrt{(z-a)(z-b)}} = \frac{1}{2z}(1-\frac{a}{z})^{-\frac{1}{2}}(1-\frac{b}{z})^{-\frac{1}{2}} = \sum_{k=1}^{\infty} h_{-k} z^{-k}, & |z| > b, \end{cases} \tag{15}$$

where the complex coefficient g_k and h_{-k} are determined in terms of the crack tip positions, *a* and *b*. Further, using the expansion of F(z) we can expand $\phi_1(z)$ and $\psi_1(z)$ at both interfaces as infinite Laurent series in the variable z. Thus, having expressed all six analytic functions in series form, the problem is reduced to a determining the unknown coefficients a_k, b_k, c_k, d_k, e_k and f_k such that the interface conditions (Eqs. 4-5) are satisfied.

3.0 NUMERICAL ANALYSIS and DISCUSSION

Since the stress intensity factor is characterized by the elastic stress distribution near the crack tip, it is reasonable to assume that it is a suitable parameter for predicting whether a crack will propagate or not. In view of this and only considering leading order terms it follows that the mode I stresses in the neighbourhood of crack tips *a* and *b* are respectively

$$\sigma_{yy} = \frac{1}{\sqrt{r_1}\sqrt{2l}}\left(-\frac{5}{8}\sin\frac{\theta_1}{2} + \frac{1}{8}\sin\frac{5\theta_1}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k a^{k-1}\right] + O\left(r_1^0\right),$$

where $z - a = r_1 e^{i\theta_1}\ (0 \le \theta_1 \le 2\pi)$ and (16)

$$\sigma_{yy} = \frac{1}{\sqrt{r_2}\sqrt{2l}}\left(\frac{5}{8}\cos\frac{\theta_2}{2} - \frac{1}{8}\cos\frac{\theta_2}{2}\right)\left[\sum_{k=-\infty}^{\infty} k f_k b^{k-1}\right] + O\left(r_2^0\right),$$

where $z - b - r_2 e^{i\theta_2}\ (-\pi \le \theta_2 \le \pi)$. (17)

Note that by the simple nature of the constructions of the function Y(z) the remote loading terms are not explicitly shown in (Eqs. 16-17). In fact, the remote loadings as well as the geometric and material parameters are incorporated within the expressions for the f_k's.
The numerical analysis rests in determining the unknown coefficients a_k, b_k, c_k, d_k, e_k and f_k. To achieve this, the series representation for the six analytic functions $\phi_i(z)$ and $\psi_i(z)$ (i=0,1,2) are substituted into the interface conditions. Knowing that a power series converges uniformly for any point *z* inside and on any circle about the origin which lies entirely inside its circle of convergence then, with negligible loss of accuracy, one can conveniently truncate the infinite series for a given number of terms. This naturally yields four truncated series expressions from which a set of linear algebraic equations can be obtained by comparing coefficients corresponding to the powers of *z*. In this work, 38 algebraic equations for 38 unknowns are generated by comparing coefficients of powers of *z* ranging from z^5 to z^{-4}. For convenience let us introduce the following non-dimensional parameters $M_0=((m_0+n_0)/(2\mu_1))R_0$ and $M_1=((m_1+n_1)/(2\mu_1))R_1$ describing the two imperfect interfaces and let us select the crack length to be fixed of length $l=R_0$. Here very small values of M_0 and M_1 (say M=0.01) corresponds to a debonded inclusion and large values of M_0 and M_1 (say $M\geq 100$) corresponds to a perfect bonding condition. Values in-between reflect imperfect bonding conditions. In addition, let us define a normalized mode I stress intensity factor as $K_I/K_{I\ \text{w/o inclusion}}$ at crack tips *a* and *b*. Here $K_I = \sigma_{yy}\sqrt{(2\pi r)}$ is the stress intensity factor calculated according to (Eqs. 16-17), respectively while $K_{I\ \text{w/o}}$

inclusion $=\sigma_\infty\sqrt{(\pi l)}$ is the mode I SIF for the same crack in a homogeneous material minus the inclusion.

As a check of our method, let us simulate the case when both interfaces are perfect (i.e. $M_0= M_1 = 100$) and thermal effects are ignored. Figure 2 shows the SIF as a function of μ_2/μ_1 for our method and that of Luo and Chen [10] under identical conditions. It is clear our results are consistent and in good agreement with those of [10].

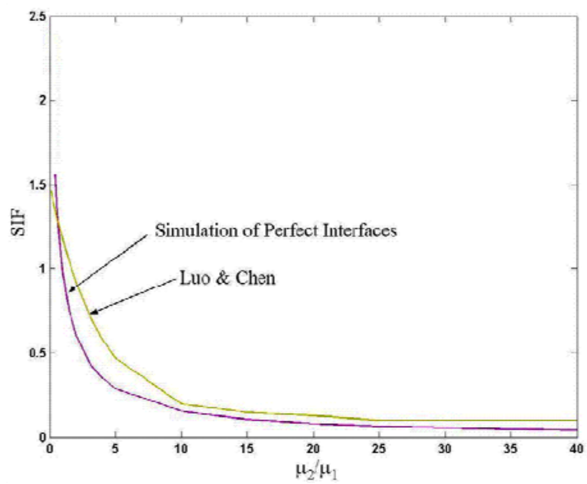

Figure2. Verification of present model

In all of the works reviewed, there has not been a discussion regarding the interaction between two imperfect interfaces and a crack. Figure 3 illustrate the changes of the normalized mode-I SIF at crack tips "*a*" and "*b*" respectively for various imperfect bonding conditions corresponding to a stiff inclusion. The results show that as the distance between the crack and the inclusion-interphase interface increases the SIF for crack tip "*a*" decreases whereas the SIF increases slightly for crack tip "*b*". This suggests that for a crack located close to the inclusion-interphase boundary there is a tendency for the crack to propagate from both crack tips. On the other hand, for highly imperfect interface conditions (say M_0=0.1), if the crack is located at some distance from the inclusion-interphase boundary, say d/R_0=2.0, then the SIF at crack tip "*a*" is less than 1 whereas the SIF at crack tip "*b*" is larger than 1 indicating that crack extension will initiate from crack tip "*b*" and move towards the surrounding matrix.

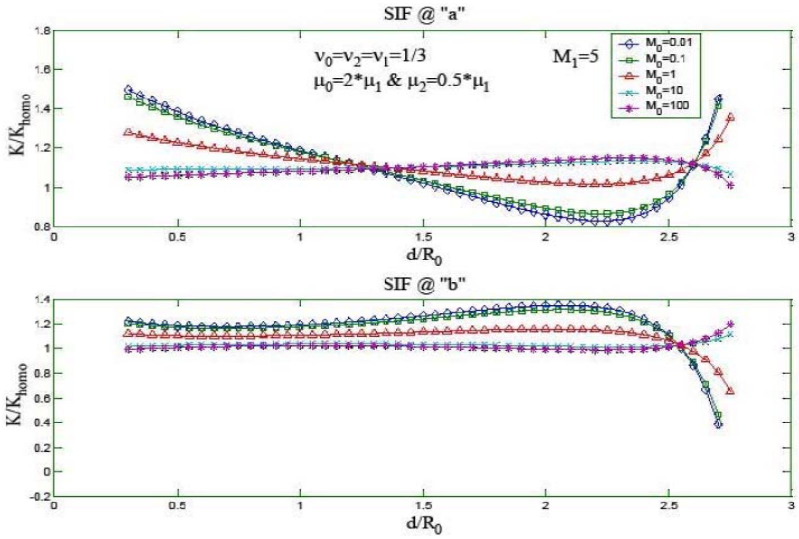

Figure 3.Influence of two imperfect interfaces (Γ_0 and Γ_1) and crack position on the normalized SIFs

The effect of material stiffness, in particular interphase layer stiffness, has a significant effect on crack behavior. Figure 4 illustrates the effect of interphase stiffness and bonding conditions at the interphase-inclusion interface on crack behaviour. The results show that the normalized SIF values increase as the stiffness of the interphase layer increase. This suggests that a stiff interphase layer has a tendency to promote crack extension from both crack tips.

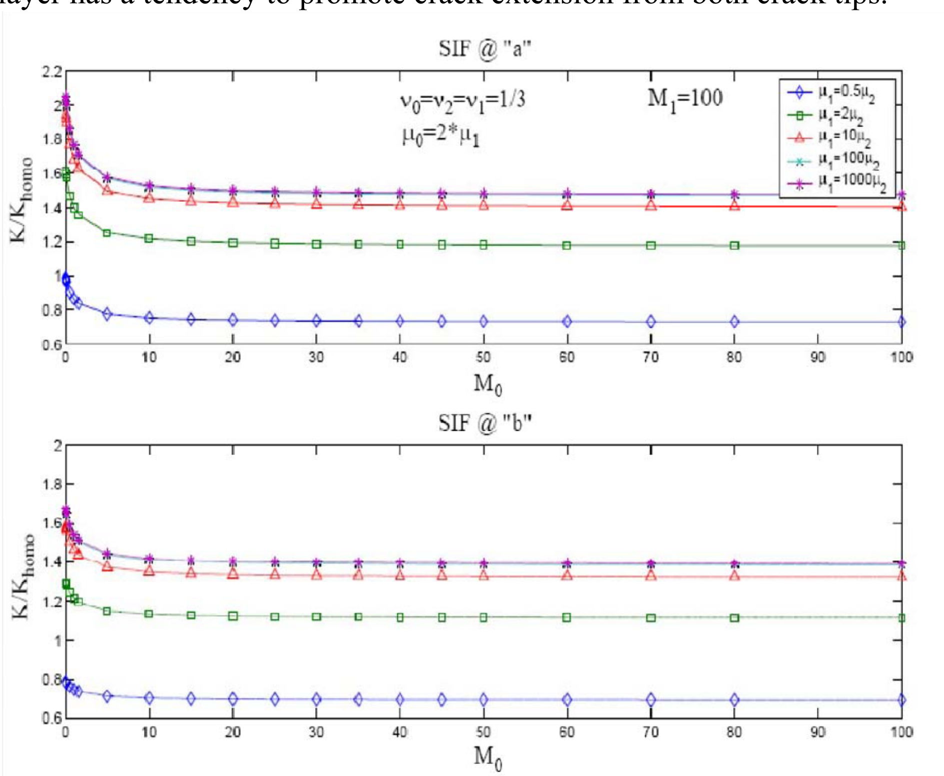

Figure 4. Influence of interphase layer stiffness and imperfect interface (Γ_0) on the normalized SIF for fixed crack position d/R_0=0.40

4.0 CONCLUSIONS

Since the extension of the dislocation density method to study three phase inclusion-crack interaction problems with imperfect interface is difficult and numerically challenging, a simple semi-analytic solution to the interaction between a pre-existing radial crack and a three-phase circular inclusion with two imperfect interfaces is presented. The numerical results clearly reveal the significance of two imperfect interface conditions and their impact on the stress intensity factor for a crack.

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