Boundary value problems posed in terms of stress orientations for plane crack systems

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1 INTODUCTION

This study presents a boundary value problem (BVP) formulated in terms of principal directions given on surfaces of curvilinear cracks in a plane.

Tractions on crack surfaces are usually assigned in formulations of boundary value problems for plane cracks [1]. This does not meet difficulties for open cracks but for bodies subjected to compressive or shear loading crack surfaces may be in full or partial contact. The latter substantially complicates formulation of conventional BVPs due to a-priory unknown positions of contact zones.

An alternative approach is suggested to deal with partly closed cracks; it assumes principle directions on cracks contours while stress magnitudes remain unknown. Such types of BVPs have been reported for simply connected domains bounded by closed contour [2-3] but not studied for open contours so far.

General formulation for a system of curvilinear cracks is presented. The method of integral equations is adopted that investigates solvability and provides a basis for numerical analysis.

It is shown that BVPs posed in terms of stress orientations has several linearly independent solutions, their number is determined by behavior of principal directions on the crack contours

2 SINGULAR INTEGRAL EQUATION OF THE PROBLEM

2.1 Problem formulation

General solution of plane problems in elasticity is given by complex potentials Φ(z) and Ψ(z) of complex variable $z=x+iy$ via the Kolosov-Muskhelishvili formulas [4] as follows(no body forces)

$$\frac{\sigma_{11}+\sigma_{22}}{2} \equiv P(z,\bar{z}) = \Phi(z)+\overline{\Phi(z)}, \quad \frac{\sigma_{22}-\sigma_{11}}{2}+i\sigma_{12} \equiv D(z,\bar{z}) = \bar{z}\Phi'(z)+\Psi(z)$$
$$2G(u_1+iu_2) \equiv W(z,\bar{z}) = \kappa\varphi(z) - z\overline{\varphi'(z)} - \overline{\psi(z)}, \quad \varphi'(z)=\Phi(z), \quad \psi'(z)=\Psi(z) \qquad (1)$$

Here σ_{jk} (j,k=1,2) are the components of symmetric stress tensor and u_k (k=1,2) are displacement components in a Cartesian coordinate frame Ox_1x_2; $z=x_1+ix_2$ is complex variable; P is a harmonic function presenting mean stress, D is the stress deviator (complex valued function); G is the shear modulus, $\kappa=3-4\nu$ for plane strain and $\kappa=(3-\nu)/(1+\nu)$ for plane stress, ν is Poisson's ratio.

The following relationships between principal stresses $\sigma_1 \geq \sigma_2$ and principal directions φ (the angle between direction of σ_1 and the real axis) determine stress deviator

$$D(z,\bar{z}) = \tau_{\max}(z,\bar{z})e^{i\alpha(z,\bar{z})},$$
$$|D| \equiv \frac{\sigma_1 - \sigma_2}{2} = \tau_{\max}(z,\bar{z}) \geq 0, \quad \arg D \equiv \alpha(z,\bar{z}) = \pi - 2\varphi(z,\bar{z}) \tag{2}$$

Let Γ be an open contour (or a set of open contours $\Gamma = \bigcup_{k=1}^{N} \Gamma_k$, k=1…n) in infinite plane. Boundary values of stresses, displacements, stress functions and complex potentials on upper/lower surfaces of Γ are designated by signs +/- receptively further on. It is assumed that stress vector $N+iT$ is continuous across Γ , thus

$$N^+(\zeta) + iT^+(\zeta) = N^-(\zeta) + iT^-(\zeta), \;\; N^\pm(\zeta) + iT^\pm(\zeta) = P^\pm(\zeta) + \frac{\overline{d\zeta}}{d\zeta}\overline{D^\pm(\zeta)}, \;\; \zeta \in \Gamma \tag{3}$$

Eqn (3) presents the first set of boundary conditions on Γ; the second set of conditions assumes known principal directions on upper/lower surfaces of Γ. Taking into account relationships (2) one can present these conditions in the form

$$\mathrm{Im}\left[e^{-i\alpha^\pm(\zeta)} D^\pm(\zeta)\right] = 0, \quad \zeta \in \Gamma \tag{4}$$

Boundary conditions (3) and (4) should be complemented by condition of single-valuedness of the displacements.

$$\int_{\Gamma_k} g(\zeta)\, d\zeta = 0, \quad g(\zeta) = \frac{d}{d\zeta}\left(W^+(\zeta) - W^-(\zeta)\right) \tag{5}$$

It is important to note that no information on stress magnitudes is present in conditions (3)-(5). Therefore, this boundary value problem belongs to the class of ill-posed problems that may possess non-unique solutions [5] vanishing at infinity.

Conditions at infinity should be specified if one seeks non-vanishing solutions. These conditions could be given in terms of stresses or in terms of principal directions, the latter means that the angle α in (2) tends to a constant α_0 when z tends to infinity.

2.2 Complex SIE

It follows from (1) and (3) that complex potentials have the form

$$\Phi(z)=\Phi_0+\frac{1}{2\pi i}\int_\Gamma \frac{g(t)}{t-z}\,dt,\quad \Psi(z)=\Psi_0-\frac{1}{2\pi i}\int_\Gamma \frac{\overline{g(t)}}{t-z}\,d\bar{t}-\frac{1}{2\pi i}\int_\Gamma \frac{\bar{t}g(t)}{(t-z)^2}\,dt \quad (6)$$

where Φ_0 and Ψ_0 are constants, which can be considered as known if homogeneous stresses are given at infinity or only arg(Ψ_0) is known if principal directions are specified at infinity. Presentation (6) already satisfies the condition of continuity of the stress vector given by (3). From (6) the expression for the stress deviator is as follows

$$D(z,\bar{z})=\Psi_0-\frac{1}{2\pi i}\int_\Gamma \frac{\overline{g(t)}}{t-z}\,d\bar{t}-\frac{1}{2\pi i}\int_\Gamma \frac{(\bar{t}-\bar{z})g(t)}{(t-z)^2}\,dt \quad (7)$$

Boundary values of stress deviator are found by the Sokhotski-Plemelj formulas [5]

$$D^{\pm}(\zeta)=\Psi_0\mp\frac{d\bar{\zeta}}{d\zeta}\mathrm{Re}(g(\zeta))-\frac{1}{2\pi i}\int_\Gamma\left(\frac{\overline{g(t)}}{t-\zeta}\frac{d\bar{t}}{dt}+\frac{(\bar{t}-\bar{\zeta})g(t)}{(t-\zeta)^2}\right)dt \quad (8)$$

Expression (8) and boundary conditions (4) allows one to form the following system of singular integral equations on Γ

$$\begin{cases}\tau^{+}_{\max}e^{i\alpha^{+}}-\tau^{-}_{\max}e^{i\alpha^{-}}=-2e^{-2i\theta}\mu\\ \tau^{+}_{\max}e^{i\alpha^{+}}+\tau^{-}_{\max}e^{i\alpha^{-}}=2\Psi_0-\mathbf{S}\left((e^{-2i\theta}+e^{-2i\beta})\mu\right)-i\mathbf{S}\left((e^{-2i\theta}-e^{-2i\beta})\nu\right)\end{cases} \quad (9)$$

Here the following notations are used $g(t)=\mu(t)+i\nu(t)$, $\beta=\beta(t,\zeta)=\arg(t-\zeta)$, $2\theta(t)=\frac{dt}{d\bar{t}}$ and the singular operator $\mathbf{S}$(g) has been introduced for compactness of the presentation

$$\mathbf{S}(g)=\frac{1}{\pi i}\int_\Gamma \frac{g(t)}{t-\zeta}\,dt,\quad \zeta\in\Gamma \quad (10)$$

It is worse to note that the second term in the second equation in (9) is regular because 2β tends to 2θ when t tends to ζ.

Boundary values of maximum shear stresses can be excluded from (9) by solving the first equation $\tau_{\max}^{\pm} = 2\mu\sin(\alpha^{\mp} + 2\theta)\left(\sin(\alpha^{+} - \alpha^{-})\right)^{-1}$, which results in the following singular integral equation (SIE)

$$2\mu\frac{e^{i\alpha^{+}}\sin(\alpha^{-}+2\theta)+e^{i\alpha^{-}}\sin(\alpha^{+}+2\theta)}{\sin(\alpha^{+}-\alpha^{-})}+\mathbf{S}\left((e^{-2i\theta}+e^{-2i\beta})\mu\right)+ \\ +i\mathbf{S}\left((e^{-2i\theta}-e^{-2i\beta})\nu\right)=2\Psi_0 \tag{11}$$

Complex SIE (11) is equivalent to system (9) if $\sin(\alpha^{+} - \alpha^{-}) \neq 0$.

2.3 Solvability of SIE (11)

Solvability of SIE (11) is determined by its dominant part and depends on the index of the coefficient of the corresponding Rieman problem that is found as follows

$$G(\zeta)=\frac{e^{i\alpha^{+}(\zeta)}\sin\left(\alpha^{-}(\zeta)+2\theta(\zeta)\right)}{e^{i\alpha^{-}(\zeta)}\sin\left(\alpha^{+}(\zeta)+2\theta(\zeta)\right)},\quad \zeta\in\Gamma \tag{12}$$

Open contour Γ can be complemented to form an arbitrary close (not self intersecting) contour by putting $G(x)$=1 on the parts connecting the rigth end of Γ_k with the left end of Γ_{k+1}. After this operation the index of a function is determined as its increment after complete traverse of the complemented contour counterclockwise divided by 2π. From (12) it is evident that the index of $G(\zeta)$ on Γ depends only on the difference of principal directions, therefore

$$\operatorname{Ind} G=\left.\frac{\alpha^{+}(\zeta)-\alpha^{-}(\zeta)}{2\pi}\right|_{\Gamma}=-\frac{1}{\pi}\left.\left(\varphi^{+}(\zeta)-\varphi^{-}(\zeta)\right)\right|_{\Gamma}=2\mathrm{K} \tag{13}$$

However the crack ends represent the points of discontinuity of $G(\zeta)$, which also contributes into the total index of the problem. This becomes evident if the asymptotic behaviour of D at the crack tips is considered. Independent of the load it can be written in the form

$$\sqrt{2\pi r}D=\left(K_{I,k}^{\pm}+3iK_{II,k}^{\pm}\right)e^{-i\vartheta/2}-\left(K_{I,k}^{\pm}-iK_{II,k}^{\pm}\right)e^{-5i\vartheta/2} \tag{14}$$

where $K_{I,k}$ and $K_{II,k}$ are stress intensity factors at the *k-th* crack end, the indices "±" refer to the right and left crack ends respectively and the angle ϑ is the polar angle in local coordinate system (r,ϑ) with the origin at the crack end. Eq (14) shows that $arg(D)$ does not depend on $K_{I,k}$ and can be determined as follows

$$\alpha_k^{\pm} = \arg K_{II,k}^{\pm} = \frac{\pi}{2}\left(1 - \operatorname{sgn} K_{II,k}^{\pm}\right) \quad \Rightarrow \quad e^{2i\alpha_k^{\pm}} = 1, \quad k = 1\ldots n \tag{15}$$

From (15) it is evident that $arg(D)$ gains the increment of $\pi/2$ after passing the crack end. Thus, the coefficient of the Riemann problem $G(\zeta)$ for the closed contour has discontinuities at crack ends and satisfies the Hölder condition everywhere except these points. The total index is further calculated by summing the index due to rotations of the principal directions (13) on the crack surfaces and the index due to discontinuity of $G(\zeta)$ at the crack ends. The latter adds unity for every crack. Therefore for n cracks the total index of the problem is 2K+n, which indicates that the solution of (11) may, in general, include a polynomial of 2K+n degree with 2K+n +1 arbitrary complex coefficients (details in [6]). However, n conditions (5) reduces the number of arbitrary constants to 2K+1. Thus, the inequality 2K ≥0 presents the condition of solvability of the problem for n cracks. The the solution of the dominant equation will in general include a polynomial of 2K+n order with 2K+1 arbitrary coefficients. This means that up to 2K+1 complex constants or (if each complex constant is counted as 2 real constants) 4K+2 real constants are included into the solution. For any negative index, 2K <0, no bounded solutions exist.

This analysis has to be acknowledged in numerical implementation. Thus, after discretisation of (19) followed, for instance, by the collocation technique, the system for the determination of unknowns should have less rank then the number of unknowns (provided that 2K≥0), which means that 4K+2 real parameters cannot be determined.

3 DEGENERATED CASES OF RECTILINEAR AND CIRCUMFERENTIAL CRACKS

In some cases the operator **S**(…) is radically simple and the analysis of solvability should be revised. Such cases include simple geometries and/or special cases of loads in which normal stresses on the cracks do not violate boundary conditions (4) and the potential Φ(z) can be determined with certain arbitrariness. Thus, the complete problem of stress tensor determination is underspecified and has infinite number of solutions.

Let Γ be a crack lying on the real axis, then $\bar{\zeta} = \zeta = x$ and $e^{-2i\theta}=1$, $e^{-2i\beta}=1$, which immediately results in

$$\mu\frac{\sin(\alpha^{+}+\alpha^{-})+2i\sin\alpha^{-}\sin\alpha^{+}}{\sin(\alpha^{+}-\alpha^{-})}+\mathbf{S}(\mu)=\Psi_0 \qquad (16)$$

In the case $\alpha^{+}+\alpha^{-}\neq 0,\pm\pi$ particular non-trivial solution of (16) can be obtained by separation of real and imaginary parts of (16), it has the following form

$$\mu(\zeta)=\operatorname{Re}\Psi_0\frac{\sin(\alpha^{+}(\zeta)-\alpha^{-}(\zeta))}{\sin(\alpha^{+}(\zeta)+\alpha^{-}(\zeta))} \qquad (17)$$

Solution (17) is valid if the following conditions are satisfied

$$\frac{1}{\pi}\int_{\Gamma}\frac{\sin(\alpha^{+}(t)-\alpha^{-}(t))}{\sin(\alpha^{+}(t)+\alpha^{-}(t))}\frac{dt}{t-\zeta}=\frac{2\sin\alpha^{-}(\zeta)\sin\alpha^{+}(\zeta)}{\sin(\alpha^{+}(\zeta)+\alpha^{-}(\zeta))}-\frac{\operatorname{Im}\Psi_0}{\operatorname{Re}\Psi_0},\quad \int_{\Gamma}\frac{\sin(\alpha^{+}-\alpha^{-})}{\sin(\alpha^{+}+\alpha^{-})}dt=0$$

In general case the problem for rectilinear crack with conditions (3)-(5) is overspecified and does not have solutions. However if $\alpha^{+}+\alpha^{-}=0,\pm\pi$ one can find the solution of (16) as follows [6]

$$\mu(x)=\cos\alpha(x)\,e^{\Lambda(x)}x^{-K}P_{2K-1}(x),\quad \Lambda(x)=\frac{1}{2\pi i}\int_{\Gamma}\frac{\ln\left(-t^{-2K}e^{-2i\alpha(t)}\right)}{t-x}dt \qquad (18)$$

where $\alpha(t)=\alpha^{+}(t)$ and $P_{2K}(z)$ is a polynomial of order 2K with arbitrary complex coefficients. Taking into account that the imaginary part of $\Lambda(x)$ is constant due to the following integral Prudnikov et al, [7]

$$\int_{-\infty}^{\infty}\frac{\ln|t|}{t-x}dt=2x\int_{0}^{\infty}\frac{\ln(t)}{t^2-x^2}dt=\frac{\pi^2}{2}$$

one can notice that the coefficients of the polynomial in (18) may be considered as real provided that the imaginary part of $\Lambda(x)$ in (18) is omitted.

It is evident that neither solution (17) nor solution (18) allows to identify $\Phi(z)$, because the function $\nu(t)$ remains unidentified. Therefore for the case of rectilinear cracks the density of crack opening displacements does not contribute in principal directions and therefore it can be chosen arbitrary. Similar conclusion is valid for the crack having the shape of an arc of certain radius. The analysis in this case is similar to that presented in [5] for the case of unit circle.

4 CONCLUSIONS

Solvability of the BVP formulated in terms of principal directions given on crack surfaces has been investigated for the case when the stress vector is continuous across the crack. Governing singular integral equation has been derived in the form given by Eq (11). The index of this SIE, 2K, has to be non-negative to provide existence of solutions. For any negative index no bounded solutions exist. It has been shown that solution includes a polynomial of 2K+n order, however n conditions of single valuedness of the displacements bring the number of independent real coefficients up to 4K+2, they remain to be free parameters of the solution.
In some special cases the problem can be underspecified and the problem has infinite number of solutions.

5 REREFRENCES

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