SIZE EFFECT ON CRACK ANALYSIS BY STRAIN GRADIENT MATERIAL MODEL

S. Imatani[1], K. Hatada[1] and G. A. Maugin[2]

[1]Department of Energy Conversion Science, Kyoto University, Kyoto 606-8501 Japan

[2]Laboratoire de Modélisation en Mécanique, CNRS/UPMC, Paris 75252 France

ABSTRACT

A strain gradient constitutive model is applied to crack analysis with the finite element simulation, and the size effect is examined on the stress concentration around crack tips. Governing equations involving the second gradient terms are derived, and a complete form of the strain gradient material model is developed within the framework of infinitesimal deformation theory. Here we employ a third order tensor, i.e. the second gradient of displacement, as a kinematic variable so that another third order tensor called "hyper-stress" appears in the equilibrium equation. And additional boundary conditions are also prescribed in terms of the gradient of displacement and couple stress. Such a treatment makes the problem statement much more complicated, compared with the conventional type boundary value problems. In contrast to the conventional first grade material model, however, the second gradient term enables us to describe the explicit size dependence of the mechanical response. The generalized variational principle called "Hellinger-Reissner principle" is applied to the mixed-type finite element stiffness equation, in which the displacement, the strain, and the second gradient of displacement as well are variants. The second part deals with the numerical simulation. The stress-strain concentration is examined, and the emphasis is placed on the explicit scale dependence of the objective domain. And the stress relaxation behavior near the crack tip is, in general, observed for smaller crack, although the tendency of the relaxation depends on the mode of cracks. The energy release rate calculated through the conventional J-integral is no more path-independent for such scale dependent crack problems.

1 INTRODUCTION

Non-simple materials with higher gradient terms have a specific feature such that the materials reveal the explicit size dependence on the mechanical response, in which the second grade material model is a typical one. The second grade material model has a long history from 60's, for example by a pioneering work by Mindlin[1]. The studies were more or less theoretical extensions of the simple material model and pointed out the explicit size dependence of the objective domain. The higher order effect has recently been renovated by several researchers such as Aifantis[2], Fleck and Hutchinson[3] from the view of size effect in the phenomenological plasticity. However, we meet a difficulty with the continuity of interpolation functions, when solving boundary value problems within a modern manner by the finite element method. This is generally overcome by introducing the Lagranigian multiplier, e.g. Shu and Fleck[4] and Amanatidou et al.[5], in the vriational formulation, although the physical evidence may be lost.

A novel methodology of stress analysis is proposed for second-grade materials in the preent investigation. Governing equations for the second grade materials are derived from the virtual work principle, and a complete set of constitutive model is formulated, in which the second gradient of displacement is involved as an independent variable. The generalized variational principle is applied to the derivation of the finite element formulation. The displacement, strain and second gradient of the displacement are regarded as variants, with which a mixed-type finite element is obtained. The scale effect due to the gradient theory is examined in the second part. Crack analyses are carried out, where we discuss the driving force for the cracking. When a crack is smaller, i.e. the objective domain becomes smaller, the second gradient effect must not be negligible. We also discuss the path dependence and independence of the energy release rate represented by J-integral

2 GOVERNING EQUATIONS

2.1 Balance equation and boundary conditions

The strain energy function W is expressible in terms of the first grade gradient of displacement u_i, i.e. strain ε_{ij}, as well as the second grade gradient η_{ijk} as $W(\varepsilon_{ij}, \eta_{ijk})$. The second gradient variable is defined by

$$\eta_{ijk} = \frac{\partial}{\partial x_k}\left(\frac{\partial u_i}{\partial x_j}\right) . \tag{1}$$

Several definitions for the second grade variable η_{ijk} are possible and then another material model can also be developed, e.g. Georgiadis[6]. Here we assume that both ε_{ij} and η_{ijk} are quite small.

Following the virtual work principle by which the internal work by the virtual displacement field is related to the external work, we have

$$\int\left(\tau_{ij}\delta\varepsilon_{ji} + \mu_{ijk}\delta\eta_{kji}\right)dV = \int_{S_t}\delta u_i \bar{t}_i dS + \int_{S_r}\delta\left(\frac{\partial u_i}{\partial n}\right)\bar{r}_i dS \quad . \tag{2}$$

Here we limit our discussion to the static case without body force nor body couple. The symbols δu_i and $\delta(\partial u_i / \partial n)$ in the right hand side represent the virtual displacement and the virtual normal gradient of displacement. Then we have the balance equation and the boundary conditions for the second grade material model:

Balance equation;

$$\frac{\partial}{\partial x_j}\left(\tau_{ji} - \frac{\partial \mu_{kji}}{\partial x_k}\right) = 0 \quad , \tag{3}$$

Surface force boundary on a part of surface S_t;

$$t_i = \bar{t}_i = n_j\left(\tau_{ji} - \frac{\partial \mu_{kji}}{\partial x_k}\right) + n_k n_j \mu_{kji} D_p(n_p) - D_j(n_k \mu_{kji}) \quad , \tag{4}$$

Prescribed displacement boundary on the complementary surface S_u;

$$u_i = \bar{u}_i \quad , \tag{5}$$

Couple force boundary on a part of surface S_r;

$$r_i = \bar{r}_i = n_k n_j \mu_{kji} \quad , \tag{6}$$

Prescribed normal displacement gradient boundary on S_e;

$$\frac{\partial u_i}{\partial n} = \frac{\partial \bar{u}_i}{\partial n} \quad . \tag{7}$$

The symbol $D_i(.)$ stands for the differential operator in tangential direction on the boundary. At first sight the governing equations are much more complicated than those in conventional model. Notice that additional boundary conditions are prescribed when we meet a corner in the objective domain, although we neglect the effect for simplicity.

2.2 Strain gradient material model

In order to complete the governing equations, we define the constitutive equations for the stress τ_{ij} and the hyper-stress μ_{ijk}. We simply assume that they are linear on the variables ε_{ij} and η_{ijk}, respectively and that the material is isotropic. The linear elasticity equation, well-known as

Hooke's law, for the first grade material model, is expressed by

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\varepsilon_{kk}\delta_{ij} = E_{ijkl}\varepsilon_{lk} \ , \tag{8}$$

where λ and μ are Lame's constants in the infinitesimal linear elasticity model. In contrast the second term is purely related to the second gradient of displacement, and the third order tensor so-called 'hyper-stress' is also reduced from the representation theorem for isotropic tensors to

$$\begin{aligned} \tau_{ijk} &= a_1\left(2\delta_{ij}\eta_{ppk} + \delta_{ipp}\eta_{jk} + \delta_{jpp}\eta_{ki}\right) + a_2\left(\eta_{ppj}\delta_{ki} + \eta_{ppi}\delta_{jk}\right) \\ &\quad + a_3\delta_{ij}\eta_{kpp} + a_4\eta_{kji} + a_5\left(\eta_{ijk} + \eta_{jki}\right) = G_{ijklmn}\eta_{nml} \end{aligned} \ . \tag{9}$$

Here we have five material constants a_1 through a_5.

When eqns (8) and (9) are substituted into eqn (3) and the kinematic variables are expressed by displacement, we have the following balance law;

$$\begin{aligned} &\frac{\partial\sigma_{ji}}{\partial x_j} - \frac{\partial}{\partial x_j}\left(\frac{\partial\tau_{kji}}{\partial x_k}\right) = (\lambda+\mu)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} \\ &- \frac{\partial^2}{\partial x_k \partial x_k}\left[(4a_1 + 2a_2 + 2a_5)\frac{\partial^2 u_j}{\partial x_i \partial x_j} + (a_3 + a_4)\frac{\partial^2 u_i}{\partial x_j \partial x_j}\right] = 0 \end{aligned} \ . \tag{10}$$

Here we introduce a new scale parameter l governing the size effect. When we regard $l^2(4a_1+2a_2+2a_5)=\lambda+\mu$ and $l^2\ (a_3+a_4)=\mu$, this will provide a hint to specify the material parameters for the second grade material. However, there are too many parameters to identify a unique set of the parameters. For the moment we set these parameters as E=2.0x10^5, ν=0.3, a_1=2.308x10^4, a_2=1.923x10^4, a_3=1.538x10^4, a_4=6.154x10^4, and a_5=3.077x10^4 for the scale parameter l=1, where Young's modulus E and Poisson's ratio ν have the well-known relationships $E=\mu(3\lambda+2\mu)/(\lambda+\mu)$ and $\nu=\lambda/2(\lambda+\mu)$.

3 FINITE ELEMENT FORMULATION

The Hellinger-Reissner (H-R) principle states that either stress or strain can be eliminated from the Hu-Washizu (H-W) principle, in which all the variables such as displacement, strain, force, and stress are variants. The H-R principle is achieved by the substitution of stress-strain relationship into the H-W principle, and therefore the displacement and the strain are now variants in the present investigation. The virtual work principle is rewritten, without loss of generality, as

$$\begin{aligned} \delta\Pi &= \int E_{ijkl}\delta\varepsilon_{lk}\,\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)dV + \int E_{ijkl}\varepsilon_{lk}\,\frac{1}{2}\left(\frac{\partial \delta u_j}{\partial x_i} + \frac{\partial \delta u_i}{\partial x_j}\right)dV \\ &+ \int G_{ijklmn}\delta\eta_{nml}\,\frac{\partial}{\partial x_i}\left(\varepsilon_{kj}\right)dV + \int G_{ijklmn}\eta_{nml}\,\frac{\partial}{\partial x_i}\left(\delta\varepsilon_{kj}\right)dV - \int E_{ijkl}\varepsilon_{lk}\delta\varepsilon_{ji}dV \\ &- \int G_{ijklmn}\eta_{nml}\delta\eta_{kji}dV - \int_{S_t}\bar{t}_i\delta u_i dS - \int_{S_r}\bar{r}_i\delta\left(\frac{\partial u_i}{\partial n}\right)dS = 0 \end{aligned} \ . \tag{11}$$

Here we introduce a new variable γ_{ij} as the first gradient of displacement $\gamma_{ik} = \partial u_i / \partial x_k$. In practice the variable γ_{ij} is expressed by way of strain ε_{ij} in the finite element equation, and so we just use the symbol as ε_{ij}.

In the variational equation, we have the displacement, the strain and the second gradient of displacement as basic variants. In such a case the choice of the order in the interpolation function is important under the mixed type finite element method. One of the possible and simple choice for the two dimensional case is as follows; the second gradient of displacement is constant in the element, the strain is interpolated by a three-node linear element, and the displacement is composed of four-node so-called “bubble” element. Let these variables be expressed by matrix form as

$$\{\square\} = [L]\{\square^l\}\ ,\quad \{\square\} = [M]\{\square^n\}\ ,\quad \{u\} = [N]\{u^n\}\ . \tag{12}$$

In the finite element scheme, the first order derivative of the variables are permissible, and so we can also write the strain in terms of displacement and the second gradient of displacement with strain as

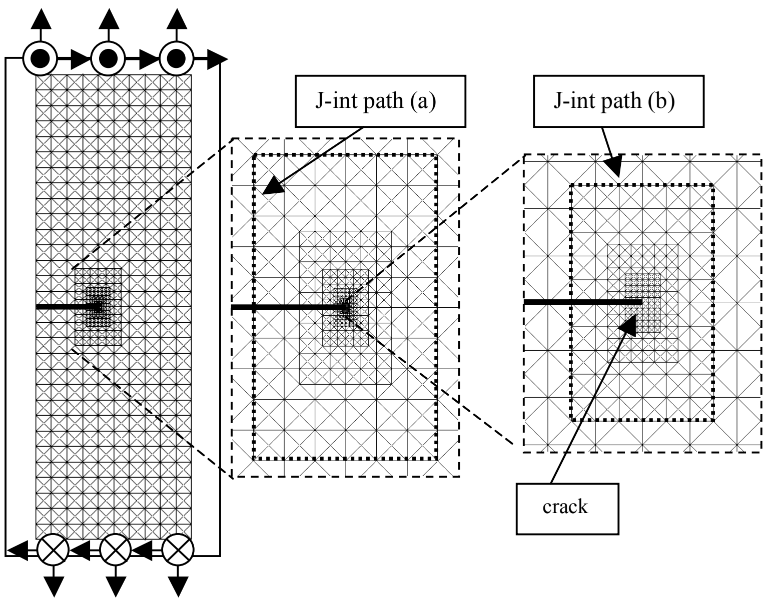
$$\{\square\} = [\square_N]\{u^n\}\ ,\quad \{\square\} = [\square_M]\{\square^n\}\ . \tag{13}$$

The second relation is used for the gradient operator on $\square_{ij}$. Using these relationships and expressing eqn (11) with matrix form, the final equations for the finite element stiffness takes the form of

$$\begin{aligned}
&\square[L]^T[G][\square_M]\{\square^n\}dV \,\square\, \square[L]^T[G][L]\{\square^l\}dV = \{0\} \\
&\square[M]^T[E][\square_N]\{u^n\}dV + \square[\square_M]^T[G][L]\{\square^l\}dV \\
&\quad \square\, \square[M]^T[E][M]\{\square^n\}dV \,\square\, \square_{S_r}[M_n]^T\{\bar{r}\}dS = \{0\} \\
&\square[\square_N]^T[E][M]\{\square^n\}dV \,\square\, \square_{S_t}[N]^T\{\bar{t}\}dS = \{0\}
\end{aligned}\ . \tag{14}$$

4 NUMERICAL RESULTS AND DISCUSSIONS

A CT-type specimen is employed for the mode-I, -II and -III crack analysis. Figure 1 shows the finite element mesh division employed in the simulation where the total number of elements is 2376 and the number of nodes is 3621. We cover the entire domain for the analysis. Figure 1(b) and 1(c) means the magnified region around the crack tip. Here we use the parameter *a*/*l* to compare the size of the domain, in which *a* indicates the crack length under *l*=1. The dotted line in the figure implies the path calculated in the J-integral evaluation.

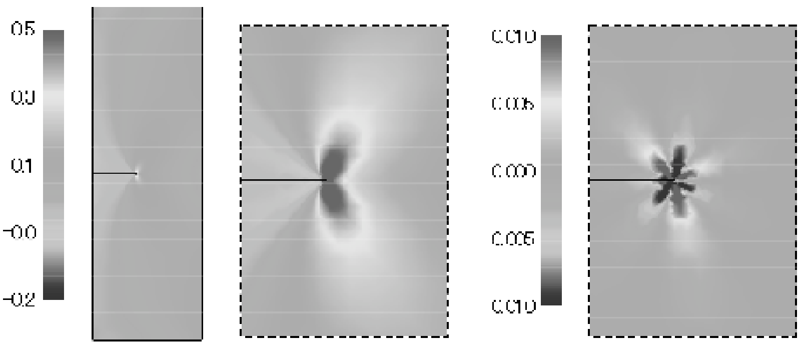


(a) Entire domain (b) Magnified (c) Crack tip

Figure 1 Finite element mesh employed in the simulation.

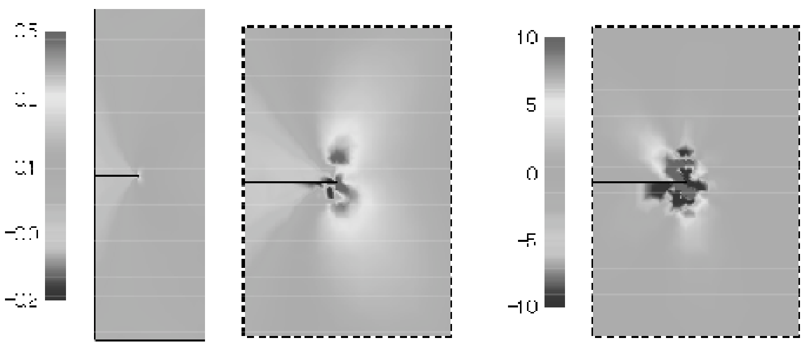
The axial strain distribution for mode-I crack is shown in Fig. 2 for the normal scale of a/l =1.0 and in Fig. 3 for the small size scale of a/l =1/1000. The right side of the figures stands for the strain just around the crack tip. The strain levels far from the crack are similar in both cases while the strain distribution is quite different at the crack tip. In particular the strain for the small domain in Fig. 3 is relaxed and the strain level is much smaller than the analysis for the normal scale, i.e. conventional analysis, in Fig. 2. The stress singularity is observed in the conventional crack problem and this kind of singularity, or irregularity, is also predicted in the small size analysis. However the quality of the singularity, i.e. the order of singularity, seems to be different from each other. The analyses for mode-II and –III cracks are also carried out, which comprise a shear of in-plane and out-of-plane deformation. We found that the stress/strain relaxation is also observed for the small object. In these cases, however, the disturbance in stress arising from the size effect is not so remarkable compared with the mode-I crack analysis.

The J-integral decreases with the decreasing size while an opposite tendency is observed in the energy release rate via stress intensity factor. When the object size is smaller, the second gradient term becomes dominant and the hyper-stress appears markedly. It implies that the strain energy is distributed into the second gradient term around the crack tip and that the J-integral composed only of first gradient underestimates the energy release rate.



(a) Whole domain ε_{yy} % (b) Near crack tip ε_{yy} % (c) Near crack tip η_{yyy} [1/mm]

Figure 2 Strain distribution for the normal scale a/l=1.



(a) Whole domain ε_{yy} % (b) Near crack tip ε_{yy} % (c) Near crack tip η_{yyy} [1/mm]

Figure 3 Strain distribution for the normal scale a/l=1/1000.

5 CONCLUSION

A second grade material model is applied to the finite element method, and the mechanical responses in crack problem are discussed. We have the following results:

(1) A rational formulation with a mixed type finite element technique is obtained, in which the Hellinger-Reissner principle is applied. The strain and the second gradient, as well as the displacement, are basic variants.

(2) The explicit size effect can be predicted by the second gradient material model. In particular the stress/strain relaxation is predicted for smaller domain, because the energy is distributed for the higher grade term in the small object.

(3) The conventional energy release rate such as J-integral must be modified to evaluate the real energy contributed to the crack intensity.

This work was initiated while the first author stayed in LMM/UPMC under the support by the French government. A part of the work was supported by the Ministry of Education, Japan, as grant-in-aid for scientific research (No. 14655044), which is also acknowledged.

REFERENCES

[1] Mindlin, R. D., Micro-structure in Linear Elasticity, Arch. Rat. Mech. Anal., 16, 51-78, 1964.

[2] Aifantis, E. C., On the Microstructual Origin of Certain Inelastic Models, J. Eng. Mat. Tech., Trans. ASME, 106, 326-330, 1984.

[3] N. A. Fleck and J. W. Hutchinson, Strain Gradient Plasticity, Adv. Appl. Mech., 33, 295-361, 1997.

[4] Shu, J. Y., King, W. E. and Fleck, N. A., Finie Elements for Materials with Strain Gradient Effects, Int. J. Num. Methods Eng., 44, 373-391, 1999.

[5] Amanatidou, E. and Aravas, N., Mixed Finite Element Formulations for Strain-gradient Elasticity Problems, Comp. Methods. Appl. Mech. Eng., 191, 1723-1751, 2001.

[6] Georgiadis, H. G., The Mode III Crack Problem in Microstructured Solids Governed by Dipolar Gradient Elasticity: Static and Dynamic Analysis, J. Appl. Mech., Trans. ASME, 70, 517-530, 2003.