STABILITY OF A DYNAMIC CRACK PROPAGATING IN AN ELASTIC STRIP

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ABSTRACT

This work comprises an asymptotic analysis of a perturbation problem for dynamic equations of two-dimensional elasticity modelling a crack propagating in an infinite strip. The sides of the elastic strip are subjected to Dirichlet boundary conditions (displacement conditions), and the crack faces are assumed to be traction-free. The unperturbed crack motion is considered as a steady, constant speed propagation along the symmetry line of the strip. Small time-dependent perturbations of the crack path include those corresponding to longitudinal variation in speed as well as deflection in the transverse direction. Asymptotic formulae for the stress intensity factors are given in terms of the “crack-path-perturbation function”, the second-order asymptotics of the weight functions and the applied load. The crack stability is analysed with respect to small perturbations of the crack path.

1 INTRODUCTION

The present work was preceded by a series of theoretical and experimental publications on fracture within thin brittle domains. These papers include the experimental and numerical work by Yang and Ravi-Chandar [1] who studied path instability for a quasi-static crack propagating in a heated strip of glass. Earlier Marder [2] analysed the elastic fields near the tip of a straight crack advancing in a thermally loaded strip and studied the variation of the Mode-I stress intensity factor K_I relative to the position of the cold front. Sasa *et al.* [3], Adda-Bedia and Ben Amar [4], Adda-Bedia and Pomeau [5] performed asymptotic analysis of quasi-static problems for two-dimensional wavy cracks and addressed the problem of crack instabilities in a heated glass strip. Liu and Marder [6] considered a model Wiener-Hopf problem associated with steady propagation of a semi-infinite crack in an elastic strip under fixed-grip loading; they have also developed an approach to the case of an accelerating crack. Crack tip instabilities in a lattice model were studied by Marder and Gross [7]. The monograph by Freund [8] presents asymptotics of elastic fields for a crack propagating in an infinite strip.

Willis and Movchan [9], [10] constructed the dynamic weight functions for a three-dimensional crack propagating in an infinite space and presented the asymptotic formulae for the stress-intensity factors for a crack with a wavy front. A two-dimensional singular perturbation problem for a moving crack was considered by Obrezanova *et al.* [11], who performed the crack path stability analysis for different types of external load.

Here we develop an asymptotic model for a crack propagating dynamically in an infinite strip under fixed-grip loading. It is assumed that the crack faces are traction-free. The unperturbed motion is considered as a steady, constant speed propagation along the line of symmetry of the strip. Small time-dependent perturbations of the crack path include those corresponding to longitudinal variation in speed as well as deflection in the transverse direction. We derive asymptotic formulae for the stress intensity factors in terms of the "crack-path-perturbation function", the second-order asymptotics of the weight functions and the applied load. We then analyse the crack stability with respect to small perturbations of the crack path.

2 FORMULATION OF THE PROBLEM

The problem considered is that of a semi-infinite crack propagating dynamically in an elastic strip $\{(x, y) : -\infty < x < \infty, \ -h < y < h\}$ of width $2h$. The material of the strip is isotropic and linearly elastic, with Lamé elastic moduli λ, μ and density ρ. The displacement field $\mathbf{u}$ satisfies the equation of motion

$$\mu \nabla^2 \mathbf{u} + (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = \rho \ddot{\mathbf{u}} \ \text{ in the strip with the crack,} \tag{1}$$

and it obeys the traction boundary conditions

$$\boldsymbol{\sigma}(\mathbf{u}) \ \mathbf{n} = \mathbf{0} \ \text{ on the crack surfaces,} \tag{2}$$

and the displacement boundary conditions

$$\mathbf{u} = \pm \ w \ \mathbf{e}^{(2)} \ \text{ on the sides of the strip } y = \pm h. \tag{3}$$

Here $\boldsymbol{\sigma}$ denotes the stress tensor, $\mathbf{n}$ is an outward unit normal, and w is a given constant.

The unperturbed configuration corresponds to a straight semi-infinite crack moving with a constant speed V along a path $M_0 = \{(x, y) : -\infty < x < Vt, \ y = 0\}$, where t denotes time. For this limit configuration, the displacement field $\mathbf{u} = \mathbf{u}^{(0)}(x - Vt, y)$ satisfies the problem (1)-(3) within the strip containing the crack M_0. The corresponding stress field is singular at the crack tip. Ahead of the crack, for $x > Vt$, the following asymptotic formula holds

$$\sigma_{22}^{(0)} \sim \frac{K_I^{(0)}}{\sqrt{2\pi(x - Vt)}} + A^{(0)}\sqrt{x - Vt} \ \text{ as } x \to Vt, \tag{4}$$

where $K_I^{(0)}$ denotes the Mode-I stress intensity factor, and $A^{(0)}$ is the coefficient which depends on the type of the applied load and the geometry of the elastic domain. As shown in [12], the coefficient $K_I^{(0)}$ has the form

$$K_I^{(0)} = (2\mu + \lambda) w \frac{\sqrt{2}}{\sqrt{\alpha h}} \frac{b^2}{a} \frac{\sqrt{R(V)}}{V}, \tag{5}$$

and $A^{(0)}$ has the same sign as $K_I^{(0)}$. In (5), a and b denote the speeds of dilatational and shear waves,

$$a^2 = (2\mu + \lambda)/\rho, \quad b^2 = \mu/\rho,$$

and

$$R(V) = 4\alpha\beta - (1 + \beta^2)^2, \quad \alpha = \sqrt{1 - V^2/a^2}, \quad \beta = \sqrt{1 - V^2/b^2}.$$

We note that the static limit (see Rice [13]) is recovered from (5) as $V \to 0$.

We then consider a small time-dependent perturbation of the crack path introduced as follows

$$M_\varepsilon = \{(x, y) : -\infty < x < Vt + \varepsilon\varphi(t),\ y = \varepsilon\psi(x)\}, \tag{6}$$

where ε is a small positive non-dimensional parameter, and φ and ψ are smooth functions characterising in-plane and out-of-plane perturbations, respectively.

The objective is to derive the asymptotic representations for the stress intensity factors for both in-plane and out-of-plane perturbations.

The reasoning of Willis and Movchan [9], and Willis and Movchan [10] applies to this first-order perturbation analysis. It requires the dynamic weight functions for the crack in a strip; these are constructed in [12].

3 ASYMPTOTIC FORMULAE FOR THE STRESS INTENSITY FACTORS

We first consider a longitudinal perturbation of the crack path. In this case the perturbation of K_I is given by

$$K_I \sim K_I^{(0)} + \varepsilon K_I^{(1)}, \tag{7}$$

and the stress intensity factor K_{II} remains zero. In the above formula (see Willis and Movchan [9])

$$K_I^{(1)} = K_I^{(0)}(Q_2(t) * \varphi(t)) + \sqrt{\frac{\pi}{2}} A^{(0)} \varphi, \tag{8}$$

where the function φ describes the longitudinal perturbation of the crack path along the x-axis (see (6)). The function Q_2 characterises the first-order perturbation of the dynamic weight function $[U_2]$ for the crack in a strip; its representation is given in [12].

For a transverse perturbation of the crack path (defined by the function $y = \varepsilon\psi(x)$ in (6)), the first-order perturbation in K_{II} has the form

$$K_{II} \sim \varepsilon K_{II}^{(1)}, \tag{9}$$

where $K_{II}^{(1)}$ is given by (see Willis and Movchan [10])

$$K_{II}^{(1)} = \sqrt{\frac{\pi}{2}}\psi(Vt)\Theta A^{(0)} + \psi'(Vt)\Upsilon K_I^{(0)} + \Theta K_I^{(0)}(Q_1(t) * \psi(Vt))$$
$$+ \left\{ [U_1] * \langle P_1^{(1)} \rangle - \langle U_2 \rangle * [P_2^{(1)}] \right\}\Big|_{X=0}. \tag{10}$$

Here

$$\Theta = \frac{4\alpha\beta - (1 + 2\alpha^2 - \beta^2)(1 + \beta^2)}{R(V)} + \frac{V^2}{b^2}\frac{(1 + \beta^2 - 2\alpha\beta)}{R(V)}, \tag{11}$$

$$\Upsilon = \frac{(1 + \beta^2)(\beta - \alpha)(\alpha + 2\beta)}{R(V)} + 2, \quad X = x - Vt. \tag{12}$$

The representations for the function Q_1 (describing the first-order perturbation of the dynamic weight function $[U_1]$ for the crack in a strip), the dynamic weight functions $[U_1]$, $\langle U_2 \rangle$ and the effective tractions $[P_1^{(1)}]$, $\langle P_2^{(1)} \rangle$ are given in [12]. The perturbation of the Mode-I stress intensity factor K_I is of order $O(\varepsilon^2)$.

4 STABILITY RESULTS

To analyse the longitudinal stability, we adopt the formula associated with the energy balance relation

$$\frac{K_I^{(1)}}{K_I^{(0)}} = \dot{\varphi}(t)\frac{\mathcal{F}'(V)}{2\mathcal{F}(V)}, \tag{13}$$

similar to Obrezanova *et al* [11]. In the above,

$$\mathcal{F}(V) = \frac{a^2 b^2 R(V)}{2\alpha(a^2 - b^2)V^2}.$$

Equation (13) is solved numerically for $\varphi(t) = e^{-i\eta Vt}$.

For the transverse stability analysis, we use the condition of "local symmetry" $K_{II}^{(1)} = 0$ and take the function ψ characterising the transverse perturbation of the crack path in the form $\psi = e^{-i\eta(X+Vt)}$.

Numerical calculations show that, below a certain speed (approximately 0.6 of the Rayleigh wave speed), the crack is unstable to both in-plane and out-of-plane perturbations. At a lower speed, the crack is likely to accelerate, initially without much deviation from straight. At a higher speed, the transverse instability becomes dominant. We would like to emphasise that these results were obtained on the basis of the Griffith energy balance and the assumption of "local symmetry". It is recognised that other physical effects, such as repeated attempts at microbranching as the crack accelerates, may be important as outlined by Sharon and Fineberg in [14], [15]. An analytical model taking into account such effects would require a separate study, perhaps including the present first-order perturbation theory as a component part.

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