RING SHAPED CRACK PROBLEM FOR A HOLLOW CYLINDER IMBEDDED IN A DISSIMILAR MEDIUM

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ABSTRACT

The analytical solution for the linear elastic problem of an internal ring-shaped crack in a transversely isotropic hollow cylinder imbedded in a transversely isotropic medium is considered. The hollow cylinder is assumed to be perfectly bonded to the surrounding medium. This structure is subjected to uniform crack surface tractions. Because of the geometry and the loading, the problem is axisymmetric. The z=0 plane on which the crack lies, is also taken to be a plane of symmetry. The composite media consisting of the hollow cylinder and the surrounding medium extends to infinity in z and r directions. The mixed boundary value problem is formulated in terms of the unknown derivative of the crack surface displacement by using Fourier and Hankel transforms. The resulting singular integral equation is solved numerically for sample cases and stress intensity factors at the circumferential crack front are presented.

1 INTRODUCTION

Structures consisting of a cylindrical cavity are commonly used in a wide variety of engineering applications. To name a few, pipes, pressure vessels, gun barrels and combustion chambers can be mentioned. Selection of the material and the determination of the dimensions under a given loading are important steps in the design of such structures. In many cases a thin or thick-walled cylinder model is quite adequate for the stress analysis. Adopting a fracture mechanics approach, however, the possibility of presence of crack like flaws should also be recognized, since the shape, size and orientation of these flaws and the disturbed stress state caused by them become critical. A number of authors have addressed such crack problems.

Erdol and Erdogan [1], for example, considered a thick walled cylinder with an axisymmetric internal or edge crack under axisymmetric loading. This problem approximates and provides a limiting case for the more realistic problem of a part-through circumferential crack lying in a plane perpendicular to the axis when the radial dimension of the flaw is relatively constant and the circumferential dimension is large compared to the wall thickness. Later Nied and Erdogan [2] considered the same geometry for the case of non-axisymmetric loads. Altınel *et al.* [3] introduced an extension in a different direction, by considering a transversely isotropic thick-walled cylinder rather than an isotropic one. Their formulation is axisymmetric and only internal cracks have been considered. In all of the above mentioned studies stress intensity factors were given. Nied [4] also considered the problem of thermal shock in a circumferentially cracked hollow cylinder with cladding. In that study, formulation was such that cladding may have different thermal properties than the base material but the structure is elastically homogeneous.

Elastically inhomogeneous cylindrical media containing cracks have also been considered by various authors. For example, the problem of an axisymmetric crack terminating at the interface of perfectly bonded, transversely isotropic, dissimilar media has been recently considered by Kadioglu [5]. In that paper, an infinitely long transversely isotropic solid cylinder containing a crack perpendicular to the axis, imbedded in a transversely isotropic medium subjected to uniform axial loading was considered. Stress intensity factors and stress distributions for sample cases were presented. Although the motivation in that paper was different, (analysing the stresses near a cracked fiber imbedded in a matrix) the formulation and the method of solution are very similar to those utilized in this study.

In this study, the analytical solution for the linear elastic problem of an internal ring-shaped crack in a transversely isotropic hollow cylinder imbedded in a transversely isotropic medium is considered. The hollow cylinder is assumed to be perfectly bonded to the surrounding medium. The problem is solved for the case of uniform crack surface tractions. Because of the geometry and the loading, the problem is axisymmetric. The z=0 plane on which the crack lies, is also taken to be a plane of symmetry. The composite media consisting of the hollow cylinder and the surrounding medium extends to infinity in z and r directions. The mixed boundary value problem is formulated in terms of the unknown derivative of the crack surface displacement by using Fourier and Hankel transforms. The resulting singular integral equation is solved numerically by using Gauss-Chebyshev quadrature formula for sample cases and the stress intensity factors at the circumferential crack fronts are presented. It is expected that this model would serve as a means to study crack problems especially for coatings inside cylindrical cavities. The coatings produced by deposition techniques sometimes have lamellar or columnar structures and this feature is taken into account by the usage of transversely isotropic materials in the model. By adopting a smeared approach, crack problems in fiber reinforced compound composite cylinders may also be addressed by this model. Further, the current formulation allows obtaining very good approximate solutions for fully isotropic materials.

2 FORMULATION OF THE PROBLEM

Geometry of the problem is given in Figure 1. It was shown in earlier studies (see for example Altınel et al. [3]) that the solution of axisymmetric elasticity problems for transversely isotropic media can be reduced to the determination of a potential function of the Love type.

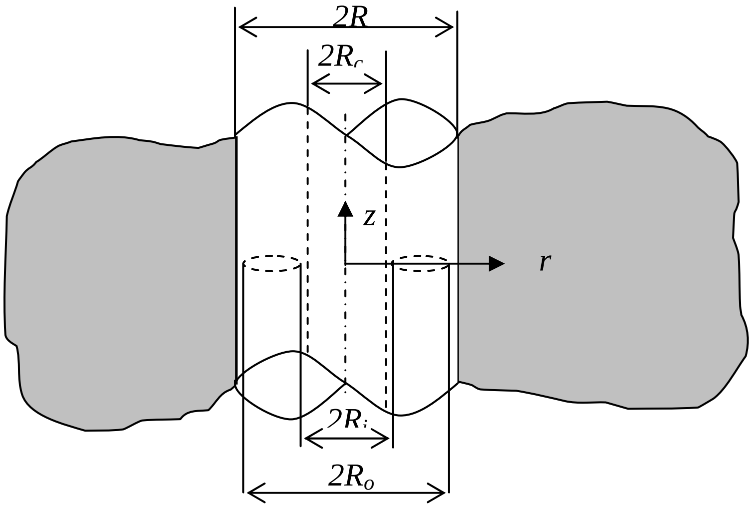

Figure 1: Geometry of the problem

The governing partial differential equation for this stress function is given as follows:

$$\frac{\partial^4\varphi}{\partial r^4}+\frac{2}{r}\frac{\partial^3\varphi}{\partial r^3}-\frac{1}{r^2}\frac{\partial^2\varphi}{\partial r^2}+\frac{1}{r^3}\frac{\partial\varphi}{\partial r}+(a+c)\frac{\partial^4\varphi}{\partial r^2\partial z^2}+\frac{(a+c)}{r}\frac{\partial^3\varphi}{\partial z^2\partial r}+d\frac{\partial\varphi^4}{\partial z^4}=0\,. \tag{1}$$

The stresses and the displacements of interest can be expressed in terms of this stress function as,

$$\sigma_{rr} = -\frac{\partial}{\partial z}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{b}{r}\frac{\partial\varphi}{\partial r} + a\frac{\partial^2\varphi}{\partial z^2}\right), \quad \sigma_{zz} = \frac{\partial}{\partial z}\left(c\frac{\partial^2\varphi}{\partial r^2} + \frac{c}{r}\frac{\partial\varphi}{\partial r} + d\frac{\partial^2\varphi}{\partial z^2}\right), \tag{2}$$

$$\tau_{rz} = \frac{\partial}{\partial r}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + a\frac{\partial^2\varphi}{\partial z^2}\right), \quad \sigma_{\theta\theta} = -\frac{\partial}{\partial z}\left(b\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r} + a\frac{\partial^2\varphi}{\partial z^2}\right), \tag{3}$$

$$u_r = -(1-b)(a_{11}-a_{12})\frac{\partial^2\varphi}{\partial r\partial z}, \quad w = a_{44}\left(\frac{\partial^2\varphi}{\partial r^2} + \frac{1}{r}\frac{\partial\varphi}{\partial r}\right) + (a_{33}d - 2a_{13})\frac{\partial^2\varphi}{\partial z^2}, \tag{4}$$

where $a = \dfrac{a_{13}(a_{11}-a_{12})}{a_{11}a_{33}-a_{13}^2}, b = \dfrac{a_{13}(a_{13}+a_{44})-a_{12}a_{33}}{a_{11}a_{33}-a_{13}^2}$,

$$c = \frac{a_{13}(a_{11}+a_{12})-a_{11}a_{44}}{a_{11}a_{33}-a_{13}^2}, d = \frac{a_{11}^2 - a_{12}^2}{a_{11}a_{33}-a_{13}^2}. \tag{5}$$

In eqn (5) $a_{11}, a_{12}, a_{13}, a_{33}$ and a_{44} are the compliances of the transversely isotropic material. The boundary conditions of the problem are given by the following equations, where the quantities with "tilde" refer to the surrounding medium.

$$\sigma_{rr}(R,z) = \tilde{\sigma}_{rr}(R,z), \qquad \tau_{rz}(R,z) = \tilde{\tau}_{rz}(R,z), \tag{6}$$

$$u(R,z) = \tilde{u}(R,z), \qquad w(R,z) = \tilde{w}(R,z), \tag{7}$$

$$\tau_{rz}(r,0) = \tilde{\tau}_{rz}(r,0) = 0, \tag{8}$$

$$\sigma_{zz}(r,0) = -p(r), \qquad R_i < r < R_0 \tag{9}$$

$$w(r,0) = 0, \qquad R_c < r < R_i \text{ and } r > R_0 \tag{10}$$

$$\sigma_{rr}(R_c,z) = 0, \qquad \tau_{rz}(R_c,z) = 0. \tag{11}$$

Note that eqns. (6)-(7) are the continuity conditions, eqn (8) is a symmetry condition, eqns (9)-(10) are the mixed boundary conditions and eqn (11) gives the stress free surface conditions inside the cylinder. Then the Love stress functions which satisfy the governing equation (1), and have the capacity to satisfy the boundary conditions (6)-(14) can be written as

$$\varphi(r,z) = \int_0^\infty \lambda(m_2 e^{r_2 z} + m_4 e^{r_4 z}) J_0(\lambda r) d\lambda$$

$$+\frac{2}{\pi}\int_0^\infty \left\{\frac{1}{(c_1^2 - c_2^2)\alpha^2}\left[AI_0(c_1\alpha r) + BK_0(c_1\alpha r)\right] + CI_0(c_2\alpha r) + DK_0(c_2\alpha r)\right\}\sin(\alpha z)d\alpha, \tag{12}$$

for the hollow cylinder containing the crack and,

$$\tilde{\varphi}(r,z) = \frac{2}{\pi}\int_0^\infty \left\{\frac{1}{(\tilde{c}_1^2 - \tilde{c}_2^2)\alpha^2}\tilde{B}K_0(\tilde{c}_1\alpha r) + \tilde{D}K_0(\tilde{c}_2\alpha r)\right\}\sin(\alpha z)d\alpha, \tag{13}$$

for the surrounding crack free medium. In these expressions, J_0 is the Bessel function, I_0 and K_0 are the modified Bessel functions of the first and the second kind; $m_2(\lambda)$, $m_4(\lambda)$, $A(\alpha)$, $B(\alpha)$, $C(\alpha)$, $D(\alpha)$, $\tilde{B}(\alpha)$ and $\tilde{D}(\alpha)$ are functions to be determined by using the boundary conditions, and $r_1, r_2, c_1, c_2, \tilde{c}_1, \tilde{c}_2$ are functions of material constants which are given by Maden [6]. By introducing

$$\frac{\partial w}{\partial r}(r,0) = G(r), \tag{14}$$

and using homogeneous boundary conditions and inverse Fourier and Bessel transforms, one can express the unknown functions m_2 , m_4, A, B, C, D, $\widetilde{B}$ and $\widetilde{D}$ in terms of $G(r)$ through a lengthy but straight forward procedure. Then, by substituting these functions into eqn (9) one can obtain the singular integral equation from which $G(r)$ can be solved as follows:

$$\int_{R_i}^{R_o} G(\rho)\left[\gamma^*\left(\frac{1}{\rho-r}+\frac{m(r,\rho)-1}{\rho-r}+\frac{m(r,\rho)}{\rho+r}\right)+2\rho k_2(r,\rho)\right]d\rho=-\pi p(r)\,,\quad R_i<r<R_o\,, \tag{15}$$

In eqn (15) γ^* is a function of material parameters, $m(r,\rho)$ is a function containing elliptic integrals which exhibits a logarithmic singularity at $r=\rho$ and $k_2(r,\rho)$ is a fredholm kernel, all given in Maden [6]. In order to complete the formulation, single-valuedness condition is written as follows:

$$\int_{Ri}^{Ro} G(\rho)d\rho=0\,. \tag{16}$$

Noting that $$G(r)=\frac{g(r)}{(R_o-r)^{1/2}(r-R_i)^{1/2}}\,, \tag{17}$$

where $g(r)$ is a bounded function, the normalized stress intensity factors at the crack tips can be expressed as

$$k(R_o)=-\frac{\gamma^* g(R_o)}{\sigma_0[(R_o-R_i)/2]}\,,\ k(R_i)=\frac{\gamma^* g(R_i)}{\sigma_0[(R_o-R_i)/2]}\,, \tag{18}$$

for the case of uniform crack surface tractions, i.e. $p(r)=\sigma_0$.

3 RESULTS AND DISCUSSION

For the problem under consideration there are four geometric parameters, namely, crack length, crack location, wall-thickness of the hollow cylinder and the hole radius in addition to the ten material parameters, i.e. five compliance values for each material. Clearly, it is not feasible to present all the results of a systematic investigation revealing the dependence of stress intensity factors (SIF) on all these parameters within the constraints of this study. Therefore a limited number of results are given by choosing only two material pairs, namely, cadmium cylinder-steel medium and Zirconia cylinder-Rene 41 medium and placing some emphasis on the variation of SIF's with respect to the geometric parameters. The number of geometric parameters is reduced to three by normalizing them using outer radius, R, of the cylinder. Note that cadmium can be used as a coating material on steel to prevent corrosion. Zirconia-rene 41 combination can be encountered in high temperature applications where rene 41 is a nickel based super alloy and zirconia is a ceramic. At this point it should be noted that the formulation is made for transversely isotropic materials and it becomes degenerate for isotropic materials. This problem, however can be easily overcome by introducing a small perturbation in one of the compliance values, thereby artificially rendering the material 'slightly' anisotropic.The compliace values of the sample materials are given in Table 1.

Normalized stress intensity factors are plotted in Figures 2 and 3 for the sample material pairs as a function of the wall thickness ratio, h/R where $h=R-R_c$. A central crack is considered which has a length equal to half the wall thickness. It can be seen that both material pairs exhibit a very similar trend. For equal net ligament lengths SIF at the inner tip is always greater than that at the outer tip and as the wall thickness ratio approaches zero, SIF's at both tips tend to their plane strain limits, a value near unity.

Normalized stress intensity factors for cadmium-steel combination are given in Table 2 . The numbers on the upper row in each cell show the SIF at the outer crack tip and the one on the lower

Table 1: Compliances of Sample Materials × 10^{11} [m^2/N]

Material	a_{11}	a_{12}	a_{13}	a_{33}	a_{44}
Cadmium	1.29	-0.15	-0.93	3.69	6.4
Steel	0.476	-0.143	-0.145	0.473	1.24
Zirconia	0.66	-0.219	-0.223	0.65	1.76
Rene 41	0.45	-0.15	-0.153	0.446	1.22

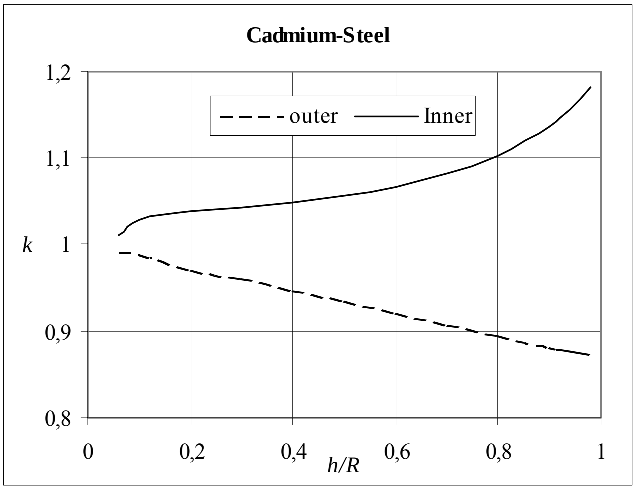

Figure 2:Normalized SIF's for cadmium hollow cylinder imbedded in steel medium

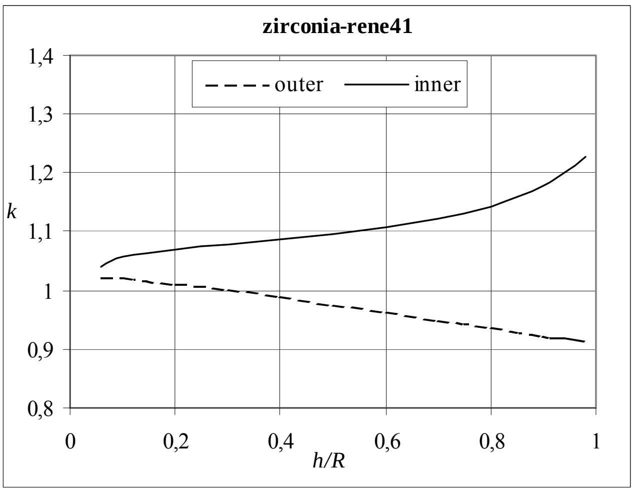

Figure 3:Normalized SIF's for zirconia hollow cylinder imbedded in rene41 medium

row show the SIF at the inner crack tip. In this table, R_c/R is kept fixed, crack length and crack location are varied. The results indicate that for this particular material combination and geometry the SIF at the inner tip is greater than that for the outer tip and the crack would propagate inward towards the free surface. It is also interesting to note that SIF do not deviate much from unity.

Table 2: Normalized Stress Intensity Factors (Cadmium -Steel, R_c/R =0.3)

R_o/R \ R_i/R	0.5	0.6	0.7	0.8	0.9
0.4	0.9869 1.0468	0.9715 1.0981	0.9482 1.1382	0.9081 1.1604	0.8316 1.1599
0.5		0.9806 1.0274	0.9604 1.0522	0.9264 1.0653	0.8555 1.0586
0.6			0.9801 1.0189	0.9513 1.0290	0.8875 1.0191
0.7				0.9780 1.0122	0.9257 1.0036
0.8					0.9676 1.0008

In Table 3 SIF's for a free standing steel hollow cylinder are given. These results are obtained by assigning very large values to the compliances of the surrounding medium. Comparison of our results (bold values in the table) with those given by Erdol [1] shows a good agreement. A comparison of results in Table 2 and Table 3 show that presence of a perfectly bonded surrounding medium significantly reduce the SIF's at both tips when the outer tip is close to the interface.

Table 3: Normalized Stress Intensity Factors for a Steel Tube (R_c/R =0.3).

R_o/R \ R_i/R	0.6		0.9	
	outer	inner	outer	inner
0.4	1.0167 **1.0138**	1.1493 **1.1494**	1.4361 **1.4220**	1.7374 **1.7374**
0.5	0.9931 **0.9917**	1.0377 **1.0390**	1.2870 **1.2769**	1.4022 **1.4083**

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