AN ELLIPTIC CRACK IN A PIEZOELECTRIC MATERIAL

T.K. Saha
Department of Mathematics, Surendranath College, Kolkata, India, E-Mail: tushar0303@yahoo.com

ABSTRACT
The three-dimensional problem of a transversely isotropic piezoelectric material with an embedded elliptic crack parallel to the material plane of isotropy is solved using an integral equation method developed earlier to solve similar problems in isotropic homogeneous elastic materials. The field equations of such medium reduce to four simultaneous partial differential equations of second order with the three displacement components and an electric potential as the basic unknowns. These partial differential equations reduce to four quasi-harmonic equations in terms of newly introduced potential functions. For the elliptic crack loaded normally with mechanical and electric loads, the mixed boundary conditions reduce the four quasi-harmonic equations to a pair of coupled integral equations – the coupling being between the normal displacement component and the electric potential function. A series of transformations together with Fourier expansions of the known and unknown fields reduce the coupled pair of integral equations to four infinite systems of Fredholm integral equations of the second kind. For constant mechanical and electric load closed form analytical solutions are obtained for the displacement and electric potentials. The solutions reveal an extremely complicated coupling between the mechanical and electric variables. The method of solution will enable one to investigate further into the fracture behavior of piezoelectric materials with main crack interacting with neighboring micro-cracks and excited not only with static loads but also with time-dependent loads. Various other problems of piezoelectric materials, e.g. vibration problems, contact problems, etc. could also be solved by this method.

1 INTRODUCTION

Piezoelectric materials have become preferred materials for a wide variety of electronic and mechatronic devices, e.g. actuators, sensors, sonar projectors, medical ultrasonic imaging applications, etc., due to their pronounced piezoelectric, dielectric and pyroelectric properties. This has increased the demand for advanced piezoelectric materials with high strength, high toughness, low thermal expansion coefficient and low dielectric constants. Piezoelectric ceramics is a kind of transversely isotropic piezoelectric material, which is adopted extensively owing to its fine piezoelectric performance. However piezoelectric ceramics are brittle and stress concentration caused by mechanical or electric loads during operation may lead to crack initiation and extension leading to failure of the components. In fact piezoelectric ceramics often possess various defects such as microcracks, microvoids, inclusions, etc., which cause geometric, electric, thermal and mechanical discontinuities and thus induce high stress and/or electric field concentrations, which may induce crack initiation, crack growth, partial discharge, and cause dielectric breakdown, fracture and failure. Due to the importance of the reliability of these devices, recently there has been tremendous interest among the scientific community in studying the fracture and failure behavior of such materials (Zhang et al. [1]; Zhang and Gao [2]).

To improve the performance and predict the reliable service life-time of piezoelectric components, it is necessary to analyze theoretically and describe quantitatively the damage and fracture processes taking place in piezoelectric materials according to the view point of coupled mechanical and electric effects. Over the last 15 years researchers have paid much attention to this field, especially with reference to piezoelectric ceramics. In the study of piezoelectric fractures and failures, the general trend is to extend the available solutions in purely elastic media to the corresponding problems in piezoelectric materials. Parton [3] probably pioneered the research in this direction by addressing the fracture problem of a through crack in a piezoelectric material.

Most available theoretical works are concerned with the two-dimensional study of cracks in piezoelectric materials. For details see the review of Zhang and Gao [2].

However piezoelectric elements in practical applications often possesses distinct geometric shapes and cracks in piezoelectric media may often be simulated as three-dimensional cracks, e.g. penny-shaped or elliptically-shaped. Hence it is essential to analyze the behavior of three-dimensional cracks in piezoelectric materials concerned with the coupled mechanical and electric effects. There are comparatively few works of three-dimensional analysis. Problems of penny-shaped cracks in transversely isotropic piezoelectric media were considered by Huang [4], Chen and Shioya [5], Kogan et al.[9] etc. But in practical situation three-dimensional cracks may be simulated more accurately by an elliptic crack. Hence effort is necessary to investigate the problems of elliptic cracks in piezoelectric materials to get a better picture of the fracture process in such materials so that the advantages of piezoelectric materials may be utilized to a greater extent. Available literatures on problems of elliptic crack in piezoelectric materials are those of Wang [6], Zi-kun and Shang-Heng [7] and Chao and Huang [8].

In earlier analysis of elliptic crack in piezoelectric material we find that either solutions are obtained as limiting case of an ellipsoidal cavity or ellipsoidal coordinate system has been used to obtain the solutions. But it may be noted that ellipsoidal coordinates are not suitable for considering finite mediums where Cartesian coordinates are more suitable. Also it is desirable to obtain the results directly instead of going to the limiting process of ellipsoidal cavity or inclusion. Further, one may expect a general method, which can take into consideration the effect of additional boundaries of a finite medium as well as the effect of nearby microcracks. Recently an integral equation method, has been developed by Roy and Chatterjee [10] to consider such problems in purely elastic homogeneous isotropic medium. The same method has been adopted to solve the title problem. Zi-kun and Shang-Heng [7] have shown that the field equations of transversely isotropic piezoelectric medium reduce to four simultaneous partial differential equations of second order with three displacement components and an electric potential as the basic unknowns which further reduce to four quasi-harmonic equations in term of newly introduced potential functions. Suitable solutions of the quasi-harmonic equations are available in term of Fourier inverse transform (Rahman [11]). The mixed boundary conditions then reduce the problem to a pair of coupled integral equations – the coupling being between the normal displacement component and the electric potential function. Then the method of Roy and Chatterjee [10] has been followed to reduce the coupled integral equations to four infinite systems of Fredholm integral equation of the second kind. For constant mechanical and electric load closed form analytical solutions are obtained for the displacement and electric potentials. The method is not only suitable for three-dimensional crack problems but also may be applied to various other problems of piezoelectric materials e.g. vibration problems, contact problems, etc.

2 POTENTIAL REPRESENTATION FOR TRANSVERSELY ISOTROPIC PIEZOELECTRIC MEDIUM

The equilibrium equations of three-dimensional piezoelectric medium are

$$\sigma_{ij,j} = 0, \tag{1}$$

$$D_{i,i} = 0, \tag{2}$$

where σ_{ij} are stress components and D_i are electric displacement components.

The piezoelectric stress constitutive relations are given by the following equations:

$$\sigma_{ij} = c_{ijkl}\varepsilon_{kl} - e_{kij}E_k, \tag{3}$$

$$D_i = e_{ikl}\varepsilon_{kl} + \varsigma_{ik}E_k, \tag{4}$$

where ε_{ij} are strain components, E_i are electric field components, c_{ijkl} are elastic stiffness constants, e_{ijk} are piezoelectric constants and ς_{ij} are dielectric constants.

The strain-displacement relations and the electric field-potential relations are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right), \tag{5}$$

$$E_i = -\phi_{,i}, \tag{6}$$

where u_i are the displacement components and ϕ is the electric potential. In equations (1) to (6) $i, j, k, l = 1,2,3.$

Now, for a transversely isotropic piezoelectric medium, introducing a Cartesian coordinate system (x, y, z) with the xOy-plane embedded in coincidence with the plane of isotropy of the medium and the z-axis perpendicular to it one can obtain the four quasi-harmonic equations following Zi-kun and Shang-Heng [7] in terms of 10 independent constants – 5 elastic, 2 dielectric and 3 piezoelectric, given below:

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + s_i \frac{\partial^2}{\partial z^2}\right)\chi_i = 0 \qquad (i = 1,2,3,4), \tag{7}$$

where χ_1, χ_2, χ_3 are three potential functions corresponding to the three roots s_1, s_2, s_3 respectively of the cubic equation

$$As^3 + Bs^2 + Cs + D = 0, \tag{8}$$

where

$$A = e_{15}{}^2 + c_{44}\varsigma_{11}, \tag{9}$$

$$B = \begin{bmatrix} 2e_{15}{}^2 c_{13} - c_{44}e_{31}{}^2 + 2e_{15}e_{31}c_{13} - 2e_{15}c_{11}e_{33} + \varsigma_{11}c_{13}{}^2 + 2c_{13}c_{44}\varsigma_{11} \\ -c_{33}c_{11}\varsigma_{11} - c_{44}c_{11}\varsigma_{33} \end{bmatrix} \div c_{11}, \tag{10}$$

$$C = \begin{bmatrix} (e_{15} + e_{31})^2 c_{33} - 2(c_{13} + c_{44})(e_{15} + e_{31})e_{33} + \varsigma_{11}c_{44}c_{33} + c_{11}e_{33}{}^2 - \\ \varsigma_{33}(c_{13} + c_{44})^2 + \varsigma_{33}\left(c_{44}{}^2 + c_{11}c_{33}\right) + 2e_{15}e_{33}c_{44} \end{bmatrix} \div c_{11}, \tag{11}$$

$$D = -\left(c_{44}e_{33}{}^2 + c_{44}c_{33}\varsigma_{33}\right) \div c_{11}, \tag{12}$$

and χ_4 corresponding to s_4 where,

$$s_4 = \frac{2c_{44}}{c_{11} - c_{12}}. \tag{13}$$

Then the general solutions of the field equations in terms of these potentials are

$$u_x = \frac{\partial}{\partial x}\left(\chi_1 + \chi_2 + \chi_3\right) - \frac{\partial \chi_4}{\partial y}, \tag{14}$$

$$u_y = \frac{\partial}{\partial y}\left(\chi_1 + \chi_2 + \chi_3\right) + \frac{\partial \chi_4}{\partial x}, \tag{15}$$

$$u_z = \frac{\partial}{\partial z}\left(k_{11}\chi_1 + k_{12}\chi_2 + k_{13}\chi_3\right), \quad (16)$$

$$\phi = \frac{\partial}{\partial z}\left(k_{21}\chi_1 + k_{22}\chi_2 + k_{23}\chi_3\right), \quad (17)$$

where k_{1j} and k_{2j} $(j = 1,2,3)$ are the three values each of k_1 and k_2 corresponding to the three roots s_j $(j = 1,2,3)$ of the equation (8) and

$$k_{1j} = \frac{s_j\left[(\varsigma_{33} - s_j\varsigma_{11})(c_{13} + c_{44}) + (e_{33} - s_j e_{15})(e_{15} + e_{31})\right]}{\left[(\varsigma_{33} - s_j\varsigma_{11})(c_{33} - s_j c_{44}) + (e_{33} - s_j e_{15})^2\right]}, \quad (18)$$

$$k_{2j} = \frac{s_j\left[(e_{33} - s_j e_{15})(c_{13} + c_{44}) - (c_{33} - s_j c_{44})(e_{15} + e_{31})\right]}{\left[(\varsigma_{33} - s_j\varsigma_{11})(c_{33} - s_j c_{44}) + (e_{33} - s_j e_{15})^2\right]}. \quad (19)$$

3 FREDHOLM'S INTEGRAL EQUATION OF THE SECOND KIND

The elliptic crack is assumed to lie in the xOy-plane with the crack center coinciding with the origin of the Cartesian coordinate system and occupying the region

$$S : \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1, \quad z = 0. \quad (20)$$

A pair of mechanical loads of identical magnitude $p(x, y)$ and in opposite directions as well as an electric load $q(x, y)$ is applied on the upper and lower surface of the crack.

The mixed boundary conditions are as follows:

$$\sigma_{xz}(x, y, 0) = \sigma_{yz}(x, y, 0) = 0 \qquad \forall(x, y), \quad (21)$$

$$\left.\begin{array}{l}\sigma_{zz}(x, y, 0) = -p(x, y)\\ D_z(x, y, 0) = q(x, y)\end{array}\right\} \quad \forall(x, y) \in S, \quad (22)$$

and,

$$\left.\begin{array}{l}u_z(x, y, 0) = 0\\ \phi(x, y, 0) = 0\end{array}\right\} \quad \forall(x, y) \notin S. \quad (23)$$

and as $\sqrt{x^2 + y^2 + z^2} \to \infty, \quad \sigma_{xz} = \sigma_{yz} = \sigma_{zz} = D_z = 0$. (24)

Because of symmetry it is sufficient to restrict attention to one half-space only, say, $z \ge 0$. Then a suitable solution of equation (7) satisfying the conditions at infinity is given by (Rahman [11])

$$\chi_j(x, y, z) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(P, Q, R, T)(\xi, \eta)\exp\left[-i(\xi x + \eta y) - m_j z\right]d\xi d\eta, \quad (25)$$

where $$m_j = \sqrt{(\xi^2 + \eta^2) \div s_j}. \quad (26)$$

Now assuming the unknown crack face displacement to be $w(x, y)$ and the unknown electric potential on the crack face to be $\Phi(x, y), \forall(x, y) \in S,$ and applying the boundary conditions following Roy and Chatterjee [10] one can obtain the following pair of coupled integral equations:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\iint_{S}\left(\xi^2+\eta^2\right)^{1/2}\begin{pmatrix} M_2 & -M_1 \\ N_2 & -N_1 \end{pmatrix}\begin{pmatrix} w(x',y') \\ \Phi(x',y') \end{pmatrix}\times$$

$$\exp\left[-i\left\{\xi(x-x')+\eta(y-y')\right\}\right]dx'dy'd\xi d\eta = 2\pi H\begin{pmatrix} -p(x,y) \\ q(x,y) \end{pmatrix} \quad \forall (x,y)\in S, \tag{27}$$

where

$$\begin{pmatrix} M_l \\ N_l \end{pmatrix} = \begin{pmatrix} B_3A_1\sqrt{s_1}-B_1A_3\sqrt{s_3} & -B_3A_2\sqrt{s_2}+B_2A_3\sqrt{s_3} \\ B_3C_1\sqrt{s_1}-B_1C_3\sqrt{s_3} & -B_3C_2\sqrt{s_2}+B_2C_3\sqrt{s_3} \end{pmatrix}\begin{pmatrix} H_{l2} \\ H_{l1} \end{pmatrix}, (l=1,2) \tag{28}$$

with $\begin{pmatrix} H_{l2} \\ H_{l1} \end{pmatrix} = \begin{pmatrix} k_{l2}B_3-k_{l3}B_2 \\ k_{l1}B_3-k_{l3}B_1 \end{pmatrix}, \quad H = H_{11}H_{22}-H_{21}H_{12},$ (29)

$$A_j = \frac{c_{33}k_{1j}+e_{33}k_{2j}}{s_j}-c_{13},$$

and, $$C_j = \frac{e_{33}k_{1j}-\varsigma_{33}k_{2j}}{s_j}-e_{31}, \qquad (j=1,2,3). \tag{30}$$

$$B_j = c_{44}\left(1+k_{1j}\right)+e_{15}k_{2j}.$$

Following Roy and Chatterjee [10] again, the coupled integral equations reduces to four infinite systems of Fredholm integral equation of the second kind written in matrix form as follows:

$\forall s = 0,1,2,\ldots,\infty,\ (n+s)\ even,\ and\ \xi, r\in[0,1]$

$$\varepsilon_s\begin{pmatrix} M_2 & -M_1 \\ N_2 & -N_1 \end{pmatrix}\begin{pmatrix} I_{s,s}{}^{c}\Psi_s(\xi) & I_{s,s}{}^{s}\overline{\Psi_s}(\xi) \\ I_{s,s}{}^{c}\Theta_s(\xi) & I_{s,s}{}^{s}\overline{\Theta_s}(\xi) \end{pmatrix} + \varepsilon_s\sum_{\substack{n=0\\ n\neq s}}\int_0^1 L_{n,s}(\xi,t)\begin{pmatrix} M_2 & -M_1 \\ N_2 & -N_1 \end{pmatrix}\times$$

$$\begin{pmatrix} I_{n,s}{}^{c}\Psi_n(t) & I_{n,s}{}^{s}\overline{\Psi_n}(t) \\ I_{n,s}{}^{c}\Theta_n(t) & I_{n,s}{}^{s}\overline{\Theta_n}(t) \end{pmatrix}dt = \pi Hb\xi^{-s}\int_0^{\xi}\frac{r^{s+1}}{\left(\xi^2-r^2\right)^{1/2}}\begin{pmatrix} -p_s(r) & -\overline{p}_s(r) \\ q_s(r) & \overline{q}_s(r) \end{pmatrix}dr, \tag{31}$$

where, $\begin{pmatrix} I_{n,s}{}^{c} & I_{n,s}{}^{s} \end{pmatrix} = \frac{1}{2}i^n(-i)^s\int_0^{2\pi}\Delta(k_0)\begin{pmatrix}\cos nX\cos sX & \sin nX\sin sX\end{pmatrix}dX$, (32)

$$\Delta(k_0) = \left(1-k_0{}^2\cos^2 X\right)^{1/2}, \qquad k_0{}^2 = \left(1-\frac{b^2}{a^2}\right), \tag{33}$$

and, $L_{n,s}(\xi,t) = (\xi t)^{1/2}\int_0^{\infty}kJ_{n+1/2}(kt)J_{s+1/2}(k\xi)dk.$ (34)

where $\Psi_n(.)\ and\ \overline{\Psi_n}(.)$ are respectively Abel's transform potentials to the Fourier cosine and sine components of the unknown displacement function $w(x,y)$ and $\Theta_n(.)\ and\ \overline{\Theta_n}(.)$ are similar terms for the unknown electric potential $\Phi(x,y)$. Also $p_s(.)\ and\ \overline{p}_s(.)$ are Fourier cosine and sine components of $p(x,y)$ and $q_s(.)\, and\, \bar{q}_s(.)$ are similar terms for $q(x,y)$ respectively.

4 SOLUTIONS FOR UNIFORM LOADING

We consider the case of constant mechanical and uniform electric loading, although solutions could be obtained for any type of polynomial loading. Thus in the present analysis we consider

$$p(x,y) = p \text{ (constant), and } q(x,y) = q \text{ (constant).} \tag{35}$$

Then equation (31) reduces to the following finite system:

$$\begin{pmatrix} M_2 & -M_1 \\ N_2 & -N_1 \end{pmatrix}\begin{pmatrix} I_{0,0}{}^c\Psi_0(\xi) \\ I_{0,0}{}^c\Theta_0(\xi) \end{pmatrix} = \pi H b \xi \begin{pmatrix} -p \\ q \end{pmatrix}, \tag{36}$$

solving which we get $\begin{pmatrix} \Psi_0(\xi) \\ \Theta_0(\xi) \end{pmatrix} = \dfrac{\pi H b \xi}{I_{0,0}{}^c\left(M_1 N_2 - M_2 N_1\right)}\begin{pmatrix} -N_1 & M_1 \\ -N_2 & M_2 \end{pmatrix}\begin{pmatrix} -p \\ q \end{pmatrix}.$ (37)

The solutions reveal that not only the material and electric constants are coupled in a complicated way but the solutions also depend on the coupled effect of the prescribed mechanical and electric loads.

In conclusion we say that the present integral equation method may be applied to such problems in finite media, dynamic crack problems, crack interaction problems and other problems of piezoelectric media e.g. vibration problems, contact problems, etc.

REFERENCES

[1] Zhang, T.Y., Zhao, M.H., Tong, P. Fracture of piezoelectric ceramics. Adv. Appl. Mech. **38**, 148-289, 2002.

[2] Zhang, T.Y., Gao, C.F. Fracture behaviors of piezoelectric materials. Theoretical and applied fracture mechanics **41**, 339-379, 2004.

[3] Parton, V.Z. Fracture mechanics of piezoelectric materials. Acta Astronaut **3**, 671-683, 1976.

[4] Huang, Jin H. A fracture criterion of a penny-shaped crack in transversely isotropic piezoelectric media. International Journal of Solids and Structures **34**, 2631-2644, 1997.

[5] Chen, W.Q., Shioya, T. Complete and exact solutions of a penny-shaped crack in a piezoelectric solid: antisymmetric shear loadings. International Journal of Solids and Structures **37**, 2603-2619, 2000.

[6] Wang, B. Three-dimensional analysis of a flat elliptical crack in a piezoelectric material. International Journal of Engineering Science **30**, 781-791, 1992.

[7] Zi-kun, W., Shang-Heng, H. Fields near elliptical crack tip in piezoelectric ceramics. Engineering Fracture Mechanics **51**, 447-456, 1995.

[8] Chao, L.P., Huang, J.H. On a piezoelectric material containing a permeable elliptical crack. International Journal of Solids and Structures **37**, 5161-5176, 2000.

[9] Kogan, L., Hui, C.Y., Molkov, V. Stress and induction field of a spheroidal inclusion or a penny-shaped crack in a transversely isotropic piezoelectric material. International Journal of Solids and Structures **33**, 2719-2737, 1996.

[10] Roy, A., Chatterjee, M. An elliptic crack in an elastic half-space. International Journal of Engineering Science **30**, 879-890, 1992.

[11] Rahman, M. Transversely isotropic solid with elliptical inclusions or cracks. Theoretical and Applied Fracture Mechanics **38**, 267-280, 2000.