DISORDER OF THE FRONT OF A TENSILE TUNNEL-CRACK PROPAGATING IN SOME INHOMOGENEOUS MEDIUM

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ABSTRACT

We study the time evolution of the shape of the front of a tunnel-crack loaded in mode I in an infinite heterogeneous medium and propagating quasistatically according to some Paris-type law. The two parts of the front are assumed to remain symmetrical and differ only slightly from straight lines at each instant, and a first-order perturbation approach is used. The geometrical disorder of the front is evaluated via the autocorrelation function of the perturbation. This disorder increases without bound, at a considerable rate, which means that the straight configuration of the front is inherently unstable in some sense. This growth rate is much larger than that found by Rice and coworkers for the problem of a semi-infinite crack propagating dynamically according to some Griffith-type law. This is due to the highly destabilizing effect of the finite crack geometry considered here. The "correlation distance" of the perturbation also increases, which mitigates the preceding conclusion since it means, in another sense, that the crack front tends to straighten back in time.

1 INTRODUCTION

Consider (Fig. 1) a tunnel-crack of width $2a$ located in an infinite isotropic elastic medium loaded by some uniform remote tensile stress σ_{yy}^{∞}. The crack is then in a situation of pure mode I, K_I being uniform along both parts of the crack front and equal to $\sigma_{yy}^{\infty}\sqrt{\pi a}$.

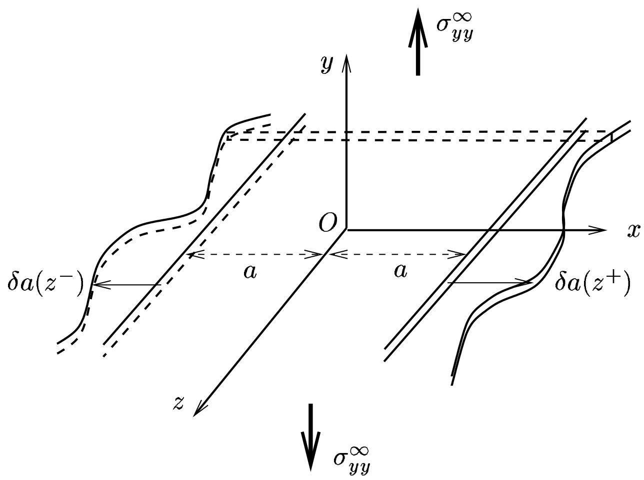

Figure 1: Tensile tunnel-crack slightly perturbed within its plane

Now slightly perturb the crack front within the crack plane; let $\delta a(z^+)$, $\delta a(z^-)$ denote the distances between the original crack front and the new one at the points $z^+ \equiv (a, 0, z)$, $z^- \equiv (-a, 0, z)$. The crack is still in a situation of pure mode I, and the variation $\delta K_I(z^\pm)$ of the stress intensity factor at the point $z^\pm$ is given, to first order in the perturbation, by

$$\begin{aligned}\frac{\delta K_I(z^\pm)}{K_I} \quad &= \quad \frac{\delta a(z^\pm)}{4a} + PV\int_{-\infty}^{+\infty} f\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^\pm)-\delta a(z^\pm)}{(z'-z)^2}\,dz' \\ &\quad + \int_{-\infty}^{+\infty} g\left(\frac{z'-z}{a}\right)\frac{\delta a(z'^\mp)}{a^2}\,dz'\,.\end{aligned} \tag{1}$$

This formula is due to Leblond *et al.* [1], who provided the functions f and g in numerical form.

Provided that some quasistatic propagation law of the crack front is specified, eqn (1) allows to study, within a first-order approach, the temporal evolution of any initial perturbation. This work considers the question of the evolution in time of the geometrical disorder of the front, propagation being assumed to be governed by some Paris-type law with fluctuating constant. It extends our previous study (Favier *et al.* [2]) of the same problem but with uniform Paris parameters[1].

In the past decade, several authors (Perrin and Rice [3], Ramanathan and Fisher [4], Morrissey and Rice [5], [6]), examined a similar problem but for a *semi-infinite* crack propagating *dynamically* according to some *Griffith-type* law. They found that the crack front disorder increased in time. The same conclusion will be shown to hold in the case investigated here, but with a much larger growth rate of disorder. The reasons for this difference will be analyzed in detail.

2 TEMPORAL EVOLUTION OF FOURIER COMPONENTS OF THE PERTURBATION

Propagation of the crack front is assumed here to be governed by some Paris-type law:

$$\frac{\partial}{\partial t}\,a(z^\pm,t) = c[1+\delta c(z^\pm,x)][K_I(z^\pm,t)]^N \tag{2}$$

where the Paris constant c fluctuates by the amount δc in the material. We also assume for simplicity that both the fluctuation of c and the perturbation of the crack front are symmetrical with respect to the Oz axis. Thus $\delta c(z^+,x) = \delta c(z^-,x) \equiv \delta c(z,x)$, and the distances from the Oz axis to the fore and rear parts of the crack front are given by

$$a(z^+,t) = a(z^-,t) \equiv a(t) + \delta a(z,t) \quad , \quad |\delta a(z,t)| \ll a(t) \tag{3}$$

at every instant, where $a(t)$ denotes the mean half-width of the crack.

We introduce the z-Fourier transforms of the fluctuation of the Paris constant and the perturbation:

$$\widehat{\delta c}(k,x) \equiv \int_{-\infty}^{+\infty} \delta c(z,x)\,e^{ikz}dz \quad , \quad \widehat{\delta a}(k,t) \equiv \int_{-\infty}^{+\infty} \delta a(z,t)\,e^{ikz}dz\,. \tag{4}$$

Inserting eqn $(4)_2$ into eqn (1), one gets

$$\frac{\delta K_I(z^\pm,t)}{K_I(t)} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left[\bar{f}(ka(t)) + \hat{g}(ka(t))\right]\widehat{\delta a}(k,t)\,\frac{e^{-ikz}}{a(t)}\,dk\,, \tag{5}$$

[1] The disorder of the front then arose solely from the assumed initial non-straightness of the front.

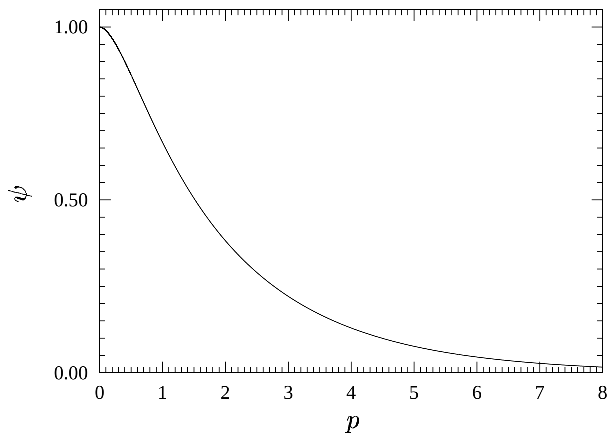

Figure 2: The function $\psi(p)$

$$\bar{f}(p) \equiv \frac{1}{4} + 2\int_0^{+\infty} f(u)\,\frac{\cos(pu)-1}{u^2}\,du \quad , \quad \hat{g}(p) \equiv 2\int_0^{+\infty} g(u)\cos(pu)\,du\,. \tag{6}$$

Inserting this result into the propagation law (2), expanding this equation to first order and identifying terms, one gets evolution equations for $a(t)$ and $\widehat{\delta a}(k,t)$. Eliminating dt between these equations, $\widehat{\delta a}$ being considered as a function of a instead of t, and integrating assuming $\delta a(z,0) = 0$, one gets the following equation, which serves as a basis for the analysis of crack front disorder:

$$\widehat{\delta a}(k,a) = \int_{a_0}^{a} \left(\frac{\psi(ka)}{\psi(ka_1)}\right)^N \left(\frac{a}{a_1}\right)^{N/2} \widehat{\delta c}(k,a_1)\,da_1\,, \tag{7}$$

$$\psi(p) \equiv \exp\left\{\int_0^p \left[\left(\bar{f}+\hat{g}\right)(p_1) - \frac{1}{2}\right]\frac{dp_1}{p_1}\right\} \tag{8}$$

where a_0 denotes the initial value of a. The function $\psi(p)$ here is represented in Fig. 2.

3 EVOLUTION OF CRACK FRONT DISORDER FOR LARGE TIMES

We consider a large number of random possible "realizations" of the heterogeneous medium and the crack geometry. Statistical invariance of the function $\delta c(z,x)$ in the directions z and x and of the function $\delta a(z,t)$ in the direction z being assumed, the two point autocorrelation functions of these functions depend only on the relative position of the points considered:

$$E[\delta c(z_1,x_1)\,\delta c(z_2,x_2)] \equiv C(z_2-z_1, x_2-x_1) \quad , \quad E[\delta a(z_1,t)\,\delta a(z_2,t)] \equiv A(z_2-z_1,t) \tag{9}$$

where $E[X]$ denotes the mathematical expectation. The functions $C(z,x)$ and $A(z,t)$ can be identified with the average values of $\delta c(z',x')\,\delta c(z'+z,x'+x)$ and $\delta a(z',t)\,\delta a(z'+z,t)$ over the crack surface and the crack front, provided an ergodic hypothesis is made. The function $A(z,t)$ and its z-Fourier transform $\hat{A}(k,t)$ (the spectral density of the perturbation) provide statistical informations about the geometry of the crack front.

3.1 *Evolution of spectral density of perturbation*

Using eqns (7) and (9), one obtains the following expression of $\hat{A}(k,t)$:

$$\hat{A}(k,t) = \int_{a_0}^{a}\int_{a_0}^{a} \frac{[\psi(ka)]^{2N}}{[\psi(ka_1)\psi(ka_2)]^N} \, \frac{a^N}{(a_1 a_2)^{N/2}} \, \hat{C}(k, a_2 - a_1) \, da_1 da_2 \tag{10}$$

where $\hat{C}$ denotes the z-Fourier transform of the function C. Careful examination of this formula reveals that for large values of t or a,

$$\hat{A}(k,t) \sim \frac{\tilde{C}(k,0)}{N|k|} \quad (k \neq 0) \quad ; \quad \hat{A}(0,t) \sim \frac{\tilde{C}(0,0)\, a_0}{N-1} \left(\frac{a}{a_0}\right)^N \tag{11}$$

where $\tilde{C}$ denotes the double z,x-Fourier transform of C. Thus, for $k \neq 0$, $\hat{A}(k,t)$ becomes asymptotically constant, and the smaller the value of $|k|$, the larger the asymptotic value of $\hat{A}(k,t)$; in contrast, $\hat{A}(0,t)$ goes to infinity. This means that the system preferentially "selects" Fourier components of $\delta a(z,t)$ with large wavelength (small $|k|$).

3.2 *Evolution of autocorrelation function of perturbation*

The autocorrelation function $A(z,t)$ of the perturbation is obtained from eqn (10) through inverse Fourier transform. Examination of the formula obtained shows that for large t,

$$A(z,t) \sim \frac{1}{2\pi} \frac{\tilde{C}(0,0)}{N-1} \left(\frac{a}{a_0}\right)^{N-1} \int_{-\infty}^{+\infty} [\psi(p)]^{2N} \, dp \, . \tag{12}$$

Thus $A(z,t)$ increases like $(a/a_0)^{N-1}$ for large t, which is a considerable rate since usual values of the exponent N (in fatigue or subcritical crack growth) are of the order of 3 to 5. In particular, taking $z = 0$, and assuming ergodicity, one sees that the standard deviation of $\delta a(z,t)$ (the square root of the mean value of $[\delta a(z,t)]^2$) increases like $(a/a_0)^{(N-1)/2}$.

3.3 *Evolution of other measures of deviation from straightness*

Other measures of deviation from straightness include the autocorrelation function of the "slope" $(\partial \delta a/\partial z)(z,t)$, and the "squared fluctuation" $E[(\delta a(z,t) - \delta a(0,t))^2]$. Both can be deduced from the autocorrelation function of $\delta a(z,t)$. Examination of the formulae obtained show that for large t,

$$E\left[\frac{\partial \delta a}{\partial z}(z_1,t)\, \frac{\partial \delta a}{\partial z}(z_2,t)\right] \sim \frac{1}{2\pi} \frac{\tilde{C}(0,0)}{N-1} \frac{1}{a_0^2} \left(\frac{a}{a_0}\right)^{N-3} \int_{-\infty}^{+\infty} [\psi(p)]^{2N} \, p^2 dp \, ; \tag{13}$$

$$E[(\delta a(z,t) - \delta a(0,t))^2] \sim \frac{1}{2\pi} \frac{\tilde{C}(0,0)}{N-1} \left(\frac{z}{a_0}\right)^2 \left(\frac{a}{a_0}\right)^{N-3} \int_{-\infty}^{+\infty} [\psi(p)]^{2N} \, p^2 dp \, . \tag{14}$$

Thus these quantities increase in time, but less quickly than $E[\delta a(z_1,t)\, \delta a(z_2,t)] \equiv A(z_2 - z_1, t)$.

3.4 Evolution of correlation length of perturbation

The "correlation length" L of the perturbation characterizes the distance over which the value of δa at some point "influences" its value at some other point. The mathematically most convenient definition of this quantity relates it to the spectral density:

$$L^2 \equiv \frac{\displaystyle\int_{-\infty}^{+\infty} \hat{A}(k,t)\, dk}{\displaystyle\int_{-\infty}^{+\infty} \hat{A}(k,t) k^2 dk} . \tag{15}$$

It is then easy to express it in terms of autocorrelation functions of the perturbation and the slope, and its asymptotic behaviour for large t follows:

$$L \sim a \sqrt{\frac{\displaystyle\int_{-\infty}^{+\infty} [\psi(p)]^{2N} \, dp}{\displaystyle\int_{-\infty}^{+\infty} [\psi(p)]^{2N} \, p^2 dp}} . \tag{16}$$

Thus L increases in proportion with the mean half-width of the crack. This means that although the disorder of the front *increases* in the sense that the amplitude of the deviation from straightness grows, it *decreases* in the sense that long-range correlations become more and more important.

4 COMPARISON WITH A SEMI-INFINITE CRACK

The case of a semi-infinite crack corresponds to $a_0 \to +\infty$, $a \equiv a_0 + \Delta a$, Δa fixed. Taking this limit in the general expressions of $\hat{A}(k,t)$ and $A(z,t)$, then letting $t \to +\infty$, one gets the following expressions of the various quantities for large times:

$$\hat{A}(k,t) \sim \frac{\tilde{C}(k,0)}{N|k|} \quad (k \neq 0) \quad ; \quad \hat{A}(0,t) \sim \tilde{C}(0,0)\, \Delta a \, , \tag{17}$$

$$A(z,t) \sim \frac{\tilde{C}(0,0)}{N\pi} \ln(\Delta a) \, , \tag{18}$$

$$E\left[\frac{\partial \delta a}{\partial z}(z_1,t)\, \frac{\partial \delta a}{\partial z}(z_2,t)\right] \sim \frac{1}{2N\pi} \int_{-\infty}^{+\infty} \tilde{C}(k,0)\, |k| \cos\left(k(z_2 - z_1)\right) dk \, , \tag{19}$$

$$E[(\delta a(z,t) - \delta a(0,t))^2] \sim \frac{1}{N\pi} \int_{-\infty}^{+\infty} \tilde{C}(k,0) \left[1 - \cos(kz)\right] \frac{dk}{|k|} \, , \tag{20}$$

$$L \sim \sqrt{\frac{2\tilde{C}(0,0)}{\displaystyle\int_{-\infty}^{+\infty} \tilde{C}(k,0)|k| dk} \ln(\Delta a)} \, . \tag{21}$$

Comparison of eqns (11) and (17), (12) and (18), (13) and (19), (14) and (20), (16) and (21) shows that all quantities, except $\hat{A}(k,t)$ for $t \neq 0$, increase more quickly for a tunnel-crack than for a semi-infinite crack. This is due to the fact that for the finite crack geometry, sinusoidal perturbations with wavelength greater than some critical value proportional to the mean half-width grow unstably (Leblond *et al.* [1], Favier *et al.* [2]); this effect disappears for a semi-infinite crack because this critical wavelength becomes infinite.

5 COMPARISON WITH THE WORKS OF RICE AND COWORKERS

The growth rate of disorder evidenced here for quasistatic propagation of a tunnel-crack obeying a Paris-type law is much stronger than that found by Rice and coworkers [3], [5], [6] for dynamic propagation of a semi-infinite crack obeying a Griffith-type law. One can show, however, that replacing Paris' law by that of Griffith can only result in an increase of the growth rate of disorder, and Morrissey and Rice [5], [6] themselves have shown that introduction of dynamic effects also enhances it, due to existence of persistent (non-decaying) "crack front waves" (first discovered by Ramanathan and Fisher [4]). The nevertheless higher growth rate of disorder found in the present work can therefore only be ascribed to existence of some characteristic length in the problem envisaged here. This conclusion is supported by the comparison presented in the previous section, although it was admittedly carried out only for Paris' law and disregarding dynamic effects.

One major motivation for this kind of study is better understanding of propagation of geological faults during earthquakes, which involves both dynamic effects (disregarded here) and existence of one or several lengthscales (accounted for here). The implication of the present work in this context is that simplifying the crack geometry by considering a model semi-infinite crack has a profound influence on the results found. The conclusion is that it would be highly desirable to consider finite cracks rather than semi-infinite ones in future studies, in spite of the technical difficulties implied.

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