ANALYSIS OF MESH DEPENDENCE IN RIGID COHESIVE INTERFACE FINITE ELEMENT MODELS

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ABSTRACT

We consider the use of initially rigid cohesive interface models in a dynamic finite element solution of a fracture process. Our focus is on convergence of finite element solutions using rigid cohesive interfaces to a continuum solution as the mesh spacing Δx (and therefore time step Δt) tends to zero. We present pinwheel meshes, which possess the "isoperimetric property" that for any curve C in the computational domain, there is an approximation to C using mesh-cell edges that tends to C including a correct representation of its length, as the grid size tends to zero. We suggest that the isoperimetric property is a necessary condition for any possible spatial convergence proof in cohesive zone modeling in the general case that the crack path is not known in advance. Conversely we establish that if the pinwheel mesh is used, the discrete interface first activated in the finite element model will converge (as the mesh size tends to zero) to the continuum initial crack. We carry out mesh refinement experiments to check convergence of both nucleation and propagation. Our preliminary results indicate that the crack path computed in the pinwheel mesh is more stable as the mesh is refined compared to other types of meshes.

1. RIGID COHESIVE MODELS

Cohesive zone modeling, which was originally proposed by Dugdale [3] and Barenblatt [1], and which was generalized and put in modern form by Rice [12], postulates that the separation of bulk material is resisted by cohesive forces governed by a constitutive model relating opening displacement to traction. Cohesive zone models are well suited for finite element analysis. The usual approach combines a volume mesh of bulk elements (to model the elastic or inelastic behavior of the undamaged material) with a network of cohesive elements that lie on some of the interfaces between the bulk elements. It is particularly natural to apply cohesive modeling to problems where defined interfaces exist. In such analyses, the fracture path is assumed *a priori* and can usually be justified by the nature of the problem (e.g., physical weak interfaces in delamination problems) or by experience gained from experiments (e.g., observations that fracture occurs frequently along inter-granular polycrystal boundaries). The interface elements are placed along the fracture path.

In applications where the crack pattern is not known in advance, notably in dynamic applications in the absence of bimaterial interfaces, interface elements cannot be prepositioned along the crack path. Instead, every edge of the bulk elements must be considered as a potential fracture surface, and the crack propagation path must be resolved as part of the solution of the governing equations. In this setting, an adaptive approach [2, 8] is often followed, in which cohesive surfaces are inserted as needed. The surface therefore is inactive prior to insertion and upon insertion is governed by the cohesive constitutive model. We call this type of model initially rigid or simply rigid (also called extrinsic [6]).

The crack path in this finite element model is composed of activated interfaces all of which lie on boundaries of cells in the initial mesh. A continuum crack path, on the other hand, has no such constraint. Allowing the crack to grow along element interfaces only, introduces obvious mesh dependence, which, if not addressed, may preclude convergence of the method as the mesh is refined. In what follows, the issue of the discrepancy between the true crack path and the discretized crack path of bulk element interfaces is considered, and a solution is proposed, namely, the pinwheel mesh. A theoretical convergence result is presented, and the paper concludes with a computational experiment.

2. REPRESENTING THE CRACK PATH IN THE MESH

To analyze the impact of mesh dependence, we must first consider the sense in which a discrete sequence of crack representations can converge to a continuum path. Note that a sequence P_i can converge to a path P in the Hausdorff sense, whereas $\mathrm{length}(P_i)$ converges to a quantity strictly greater than $\mathrm{length}(P)$. The Hausdorff distance between P_i and P is the maximum over all points $a \in P_i$ of the distance from point a to the closest point $b \in P$.An example like this is not pathological; indeed, failure of length to converge is the typical behavior for any family of structured meshes.

The length of the discrete versus continuum crack path is significant physically because the energy needed to create a crack is proportional to its length. This leads us to conjecture that a necessary condition for spatial convergence in cohesive modeling is that the sequence of meshes $\mathcal{M}_1, \mathcal{M}_2, \ldots$ must contain a sequence of paths $P_1, P_2, \ldots$ such that $P_i \to P$ in the Hausdorff sense and such that $\mathrm{length}(P_i) \to \mathrm{length}(P)$, where P is the continuum crack solution.

To provide some evidence for this conjecture, we carried out a computational simulation of the experiment used by Kalthoff and Winkler [5] to study failure mode transition. In the setup shown in Fig. 1 a plate of high strength steel with two edge notches was impacted by a steel projectile with speed v_o. Two different modes of failure were observed as v_o was varied. At low impact speeds, the plate failed in a brittle manner with a crack propagating at an angle of about $70°$ counterclockwise with respect to the original notch. Ductile failure in the form of an adiabatic shear band occurred at high impact speeds, with the shear band forming ahead of the notch at an angle of about $10°$ clockwise. We only simulated the brittle fracture mode.

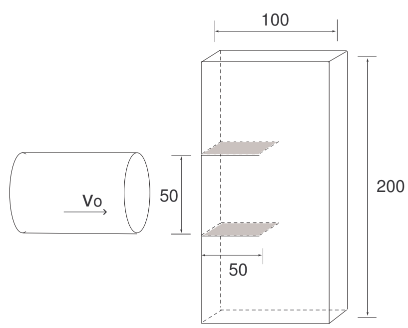
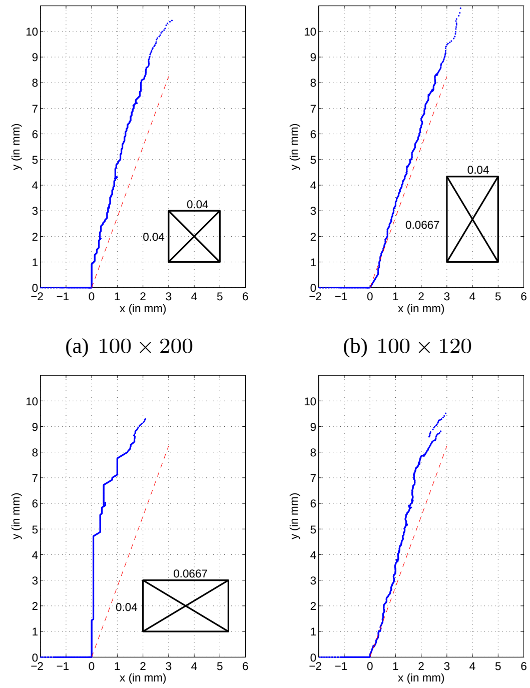


Figure 1: Schematic of the double edge-cracked specimen. Dimensions in mm.

To investigate the effect of mesh angles on the computed crack path, a $4\mathrm{mm} \times 8\mathrm{mm}$ region ahead of the initial crack tip was discretized by a structured mesh consisting of rectangular cells, each subdivided into four triangles; the rest of the domain was covered with a gradually coarsened unstructured mesh to reduce computational time. The corresponding cells' height-to-width aspect ratios are 1.67, 1 and 0.6 respectively. A complete unstructured mesh was also used for comparison. Fig. 2 shows the fracture paths obtained with the four meshes. It is obvious that the results depend on the layout of the mesh. In particular, the 100×120 mesh and the unstructured mesh seem to best match the experiment, and these also are the meshes that have the best approximation to a $70°$ crack path.

3. PINWHEEL MESHES

As mentioned earlier, the goal of rigid cohesive modeling is often to solve problems in which the crack path is not known in advance but is an outcome of the simulation. Based on the results in



(a) 100×200 (b) 100×120

(c) 60×200 (d) unstructured

Figure 2: Fracture paths obtained with meshes having different aspect ratios. The dashed reference line is 70° counterclockwise from the origin.

the previous section, we conclude that if spatial convergence is desired, the family of meshes must represent every possible crack path both in the Hausdorff sense and in the sense of length. There is one family of meshes known to have this property, namely, pinwheel meshes in two dimensions. The pinwheel mesh is derived from Radin and Conway's [10] pinwheel tiling, which we now describe. The pinwheel tiling starts with a $1 : 2 : \sqrt{5}$ right triangle T_0 and subdivides it into five $1 : 2 : \sqrt{5}$ subtriangles that are similar to each other. See Fig. 3. This subdivision process may be repeated indefinitely, yielding a tiling of the original triangle with an arbitrary level of refinement. The distinction between a tiling and a mesh is that the triangles in a tiling are not required to meet edge-to-edge and therefore may have hanging nodes.

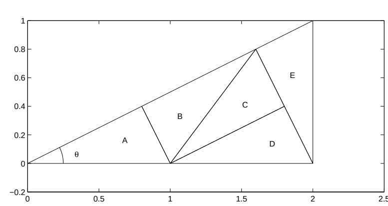


Figure 3: The basic pinwheel subdivision of a $1 : 2 : \sqrt{5}$ right triangle.

Radin and Sadun [11] have proved that the pinwheel tiling has the following isoperimetric property. Let $\mathcal{M}_1, \mathcal{M}_2, \ldots$ be the sequence of pinwheel tilings such that $\mathcal{M}_i$ repeats the construction in the previous paragraph to the ith level (and thus contains 5^i tiles). Let L be an arbitrary line segment in the initial triangle T_0. Then $\mathcal{M}_i$ contains a path L_i such that the L_i's converge to L in the Hausdorff and length sense. Their theorem extends easily to arbitrary curves, since any curve can be approximated in the two senses by a path of line segments, and then the line segments can be approximated by the mesh.

In recent work [4], we generalized the pinwheel tiling to arbitrary triangles (not only $1 : 2 : \sqrt{5}$) and showed how to use this generalization to create a mesh generator that possesses the isoperimetric property for arbitrary two-dimensional polygonal domains.

4. SPATIAL CONVERGENCE THEOREM

Rigid cohesive models defined on pinwheel meshes satisfy what we believe to be the necessary condition for spatial convergence, so we can now consider whether spatial convergence is indeed attained on these meshes. This question is far from trivial, since the presence of a path that approximates the continuum solution does not imply that a cohesive finite element model will find that path.

In this section we present a convergence result that is a first step toward a complete theory of spatial convergence. Our theorem covers only nucleation, i.e., the initial formation of a crack. Our setting for the continuum problem is as follows. We consider a two-dimensional body whose motion is governed by linear elastodynamics. We hypothesize that a crack nucleates in this body whenever the stress tensor first attains a principal value of σ_c, where $\sigma_c > 0$ is called the critical stress. We assume that the crack nucleates at the point where σ_c is first achieved. We hypothesize that its orientation is orthogonal to the dominant principal axis of the stress tensor at that point.

Our finite element model is the usual discretization of linear elasticity on triangles. We assume initially rigid cohesive interfaces at all element boundaries with an activation criterion that the normal traction on the interface reaches σ_c.

We also hypothesize that the sequence of meshes $\mathcal{M}_1, \mathcal{M}_2, \ldots$ has grid size tending to zero and is quasi-uninform. The sequence also has the following property. For every point x in the computational domain, for every angle θ, and for every $\epsilon > 0$, there is an $I > 0$ such that every mesh M_I, $M_{I+1}, \ldots$ contains an interface whose distance from x is at most ϵ and whose orientation is at most ϵ away from θ. Our pinwheel family of meshes have these properties. Note that these properties are different from, but apparently closely related to, the isoperimetric property.

Under all these assumptions plus a few more technical assumptions about smoothness, we prove that the first interface to activate in the discrete model converges to the continuum nucleation point in three senses: the location of the interface converges to the location of the continuum point, the orientation of the interface converges to the continuum nucleation orientation, and the time of nucleation converges to the continuum time. Our proof [9] is not difficult but is technical in the sense that it relies heavily on compactness arguments. Our theory does not extend to crack propagation. It would extend to three dimensions if a three-dimensional analog of pinwheel meshes could be constructed.

5. A MESH REFINEMENT STUDY

The setup of the study is of the compression compact specimen (CCS) impact-experiment used by Rittel et. el. [13, 7]. The schematic of the experiment is shown in Figure 4. The PMMA specimen is fractured by impacting with a Hopkinson bar, and the load at the bar-specimen interface can be measured and subsequently used for the boundary condition of our simulations. We investigate, through a series of simulations, the effects of different mesh layouts on the crack path. This setup is more suitable for a convergence study than the Kalthoff-Winkler experiment because the crack path is not a straight line segment, and therefore there is no structured mesh that exactly represents it.

Three mesh types—structured, unstructured, and pinwheel—are considered with several different levels of refinement repeated for each of them. Since this problem has no known analytic solution, a comparison of crack paths from different levels of refinement of the mesh seems to be the best approach for checking convergence.

The cohesive model is the initially rigid model of [14] with $\sigma_c = 105\text{MPa}$, $G_c = .0048\text{MPa-m}$, $\beta = 1$. As is apparent from Fig. 5, the pinwheel mesh at the coarse level appears to be closer to the fine pinwheel mesh than the other two families. The trajectory of the crack in the fine pinwheel mesh required substantial CPU time to compute and was not completely traced at the time this abstract was submitted. At initiation the paths found in the three pinwheel meshes are identical, thus providing computational confirmation of the theoretical convergence result, which covers crack initiation.

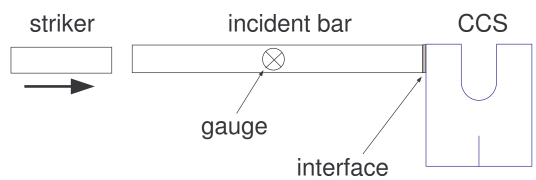


Figure 4: Schematic of compression compact specimen impact-experiment. The dimensions of the specimen are 51mm high 46mm wide, 11mm deep. The notch at the bottom is 12 mm long, and the U-shaped notch is 15mm wide and 27.5mm high.

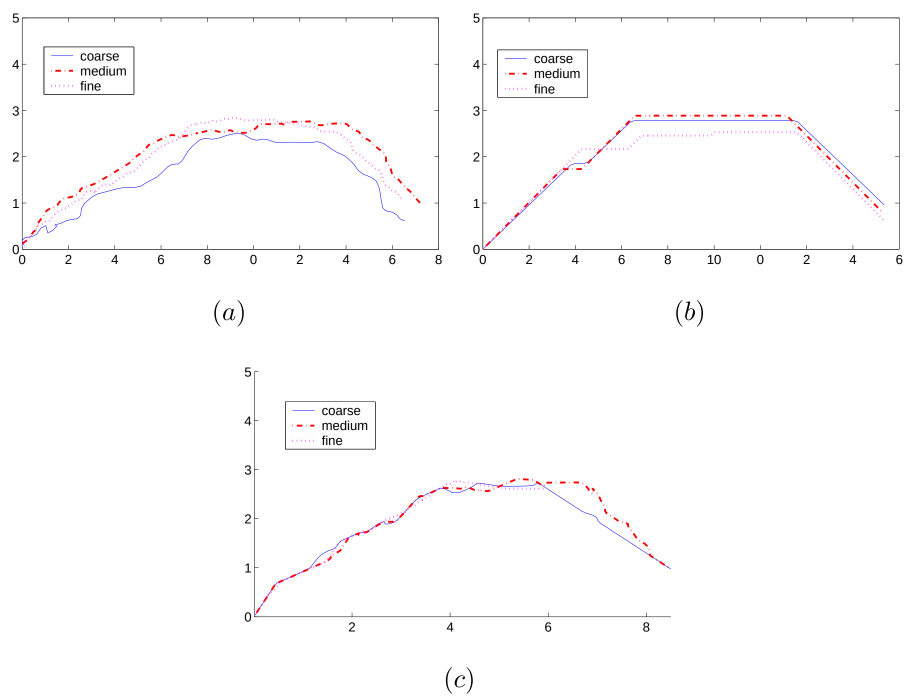


Figure 5: Refinement study using (a) an unstructured mesh, (b) a structured mesh, and (c) the pinwheel mesh.

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