HYSTERETIC FLEXURAL BEHAVIOUR OF A REINFORCED CONCRETE BEAM WITH A T CROSS-SECTION

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ABSTRACT

The behaviour of reinforced concrete (R.C.) structures under cyclic loading has been studied for only a few decades, and extended studies are more and more needed for structures such as bridges, where fatigue phenomena can be remarkable. In the present paper, a theoretical model based on fracture mechanics concepts is proposed in order to analyse the mechanical damage of a reinforced concrete beam with a T cross-section subjected to cyclic bending. Local phenomena, such as fracturing or crushing of concrete and yielding or slippage of the longitudinal steel reinforcement, are examined. Further, fatigue life is predicted by applying a crack growth law, and the energy dissipated during the plastic shake-down phenomenon is evaluated.

1 INTRODUCTION

The behaviour of ordinary or prestressed reinforced concrete (R.C.) structures under cyclic loading has been studied for only a few decades, and some Standards [1,2] give us rules on how to take into account damage phenomena when designing R.C. structures, but additional investigations are needed. Some interesting research works about fatigue behaviour of concrete can be found in Ref.[3] and in several theoretical and experimental papers (for instance, [4-10]).

In the present paper, a fracture mechanics model proposed in Refs [4-7] for ordinary R.C. beams with a rectangular cross-section under cyclic bending is extended to beams with a T cross-section. Local phenomena, such as fracturing or crushing of concrete and yielding or slippage of the longitudinal steel reinforcement, are examined. Fatigue life and the energy dissipated during the plastic shake-down phenomenon can be evaluated.

2 COMPLIANCES OF A CRACKED BEAM WITH A T CROSS-SECTION

Examine a linear elastic beam of length $2\,l$, with a T cross-section. A through-thickness edge crack of relative depth $\xi = a\,/\,b$, with $a \le (b - b_T)$, is assumed to exist in the lower part of a given beam cross-section (Fig.1). The cracked cross-section is divided into three sub-sections, A, B and C, and the beam can be modelled by springs arranged in series and parallel. Consequently F^* and M^* can be expressed as follows :

$$\begin{aligned} F^* &= F_A + F_B + F_C = S_{A,F}\; u + S_{B,F}\; u + S_{C,F}\; u = \\ &= u\left(S_{A,F} + S_{B,F} + S_{C,F}\right) \end{aligned} \tag{1}$$

$$\begin{aligned} M^* &= M_A + M_B + M_C - F_A\,\frac{b-b_T}{2} - F_C\,\frac{b-b_T}{2} = \\ &= \varphi\left(S_{A,M} + S_{B,M} + S_{C,M}\right) - u\,\frac{b-b_T}{2}\left(S_{A,F} + S_{C,F}\right) \end{aligned} \tag{2}$$

where $S_{A,F}$, $S_{B,F}$, $S_{C,F}$ are the extensional stiffnesses of the elements A, B and C, respectively, whereas $S_{A,M}$, $S_{B,M}$, $S_{C,M}$ are the bending stiffnesses of the above elements. Such stiffnesses are equal to:

$$S_{A,F} = S_{C,F} = \frac{E\;b_T}{4\;l}\,(w-t) \tag{3}$$

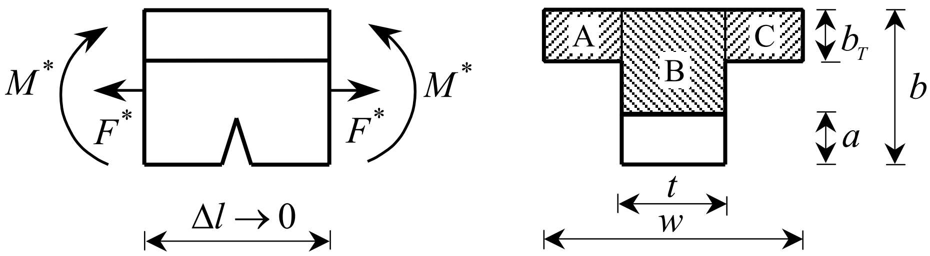

Figure 1: Cracked beam with a T cross-section.

$$S_{B,F} = 1/\lambda_{B,F} = 1/\lambda_F \tag{4}$$

$$S_{A,M} = S_{C,M} = \frac{E\ b_T^3}{48\ l}(w-t) \tag{5}$$

$$S_{B,F} = 1/\lambda_{B,F} = 1/\lambda_F \tag{6}$$

with the compliances λ_F and λ_M given by eqns (19) and (21) in chapter 18 of Ref.[11], respectively. Further, E is the Young modulus of the material, and the cross-section sizes are shown in Fig.1. Therefore, the total relative displacement u of the beam ends can be obtained from eqns (1), (3) and (4), whereas the total relative rotation φ can be computed through eqns (2), (5) and (6).

Then, the stress-intensity factor is determined :

$$K_I = \frac{F_B}{b^{1/2}\ t} Y_F(\xi) + \frac{M_B}{b^{3/2}\ t} Y_M(\xi) = \frac{S_{B,F}\ u}{b^{1/2}\ t} Y_F(\xi) + \frac{S_{B,M}\ \varphi}{b^{3/2}\ t} Y_M(\xi) \tag{7}$$

for $\xi = a/b \le 0.6$ [12] and, by substituting u and φ deduced as is described above, we get :

$$K_I = R_F(\xi)\ \frac{F^*}{b^{0.5}\ t}\ Y_F(\xi) + R_M(\xi)\ \frac{M^*}{b^{1.5}\ t}\ Y_M(\xi) \tag{8}$$

where
$$R_F(\xi) = \frac{S_{B,F}}{\sum\limits_{i=A,B,C} S_{i,F}} \tag{9}$$

$$R_M(\xi) = \frac{S_{B,M}}{\sum\limits_{i=A,B,C} S_{i,M}} \left(1 + \frac{b - b_T}{2e}\ \frac{S_{A,F} + S_{C,F}}{\sum\limits_{i=A,B,C} S_{i,F}} \right) \tag{10}$$

Now the localised compliances (λ_ℓ) of the cracked T cross-section being analysed are evaluated. The relative displacements between the ends of the infinitesimal beam portion containing the cracked cross-section (Fig.1, left) can be expressed through the following equations:

$$u_{\ell,T} = \lambda_{\ell,F,T}\ F^* + \lambda_{\ell,FM,T}\ M^* \tag{11}$$

$$\varphi_{\ell,T} = \lambda_{\ell,FM,T}\ F^* + \lambda_{\ell,M,T}\ M^* \tag{12}$$

where ℓ stands for localised, and the subscript T indicates that the behaviour of a beam with a T cross-section is being examined.

The localised extensional compliance $\lambda_{\ell,F,T}$ can be computed by letting $M^* = 0$ and assuming that all the elastic work done by F^* is used to create an increment of cracked surface :

$$\frac{1}{2}\ F^*\ \mathrm{d}\,u_{\ell,T}\ =\ \mathrm{G}\ t\ \mathrm{d}a \tag{13}$$

where the energy release rate G is related to K_I by the expression $\mathrm{G} = K_I^{\,2} / E$. By employing the relationship $\mathrm{d}a = b\ \mathrm{d}\xi$ and eqns (8) and (11) for $M^* = 0$, we can obtain from eqn(13) :

$$\lambda_{\ell,F,T}\ =\ \frac{2}{t\,E}\int_0^{\xi} R_F^2(\xi)\ Y_F^2(\xi)\ \mathrm{d}\xi\quad . \tag{14}$$

Similarly, the localised bending compliance can be obtained from the energy balance equation in the case of $F^* = 0$:

$$\lambda_{\ell,M,T}\ =\ \frac{2}{b^2\ t\,E}\int_0^{\xi} R_M^2(\xi)\ Y_M^2(\xi)\ \mathrm{d}\xi \tag{15}$$

and the combined extension and bending compliance is determined from the energy balance by taking into account both F^* and M^* :

$$\lambda_{\ell,FM,T}\ =\ \frac{2}{b\,t\,E}\int_0^{\xi} R_F(\xi)\ R_M(\xi)\ Y_F(\xi)\ Y_M(\xi)\ \mathrm{d}\xi \tag{16}$$

3 R.C. BEAM UNDER MONOTONIC BENDING

Consider a cracked R.C. beam with a T cross-section under a monotonic bending moment M (Fig.2). A rigid-perfectly plastic behaviour is assumed for steel, whereas a linear elastic constitutive law is supposed for concrete. Further, assume that the crack depth a is greater than the steel cover h , and therefore the concrete cross-section is subjected to M and the eccentric compressive axial force F due to the tensile reaction of the longitudinal steel reinforcement. Consequently, the global internal reactions on the concrete cross-section are equal to :

$$M^*\ =\ M\ -\ F\ (b/2 - h) \tag{17}$$

$$F^*\ =\ -\ F \tag{18}$$

Up to the instant when steel yields, the relative rotation (eqn(12)) of the ends of the infinitesimal beam portion (Fig.2, left) is assumed to be equal to zero [4]. Through this compatibility condition and eqns (17) and (18), the relationship between the applied bending moment M and the tensile force F of the longitudinal steel reinforcement is determined :

$$M\ =\ F\,b\left[\frac{1}{2} - \frac{h}{b} + r_T(\xi)\right]\quad ,\quad \text{with} \tag{19}$$

$$r_T(\xi)\ =\ \frac{\lambda_{\ell,FM,T}}{b\ \lambda_{\ell,M,T}}\ =\ \frac{\int_0^{\xi} R_F(\xi)\ R_M(\xi)\ Y_F(\xi)\ Y_M(\xi)\ \mathrm{d}\xi}{\int_0^{\xi} R_M^2(\xi)\ Y_M^2(\xi)\ \mathrm{d}\xi} \tag{20}$$

The steel yielding occurs when the bending moment is equal to M_P, which can be obtained from eqn(19) by substituting the force $F_P = f_y A_s$ of steel plastic flow to F , where f_y and A_s are the yield strength and the area of the longitudinal steel reinforcement, respectively. Note that the plastic

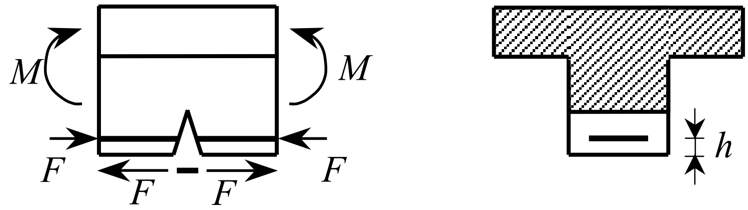

Figure 2: Cracked T cross-section of a reinforced concrete beam under bending moment M.

bending moment M_P can be computed only numerically from eqn(19) (that is, eqn(19) is not a closed form expression of M_P) since $r_T(\xi)$ is a nonlinear function of M_P : as a matter of fact, R_F and R_M are functions of $S_{B,F}$ and $S_{B,M}$ (eqns (9) and (10)), and such stiffnesses depend on λ_F and λ_M (eqns (4) and (6)), which vary with e , that is to say, with M_P :

$$e = M^* / F^* = \left[M_P - F_P \left(\frac{b}{2} - h \right) \right] \Big/ \left[-F_P \right] \quad . \tag{21}$$

Another possible damage phenomenon is that due to the slippage of the longitudinal steel reinforcement. Such a slippage occurs in correspondence to the bending moment M_{PO} , which can be computed from eqn(19) by substituting the force F_{PO} of pull-out to F . As an initial attempt, this force of steel slippage could be assumed equal to :

$$F_{PO} = \sum_{i=1}^{n} \tau_f \quad \pi \Phi_i \quad 5 \Phi_i \quad , \tag{22}$$

with n = number of longitudinal steel bars, $\pi \ \Phi_i$ = cross-section perimeter of the i-th bar, and $5\ \Phi_i$ = length of the bar portion on which the shear stress τ_f of friction between steel and concrete acts [1]. Analogous to the case of the plastic bending moment M_P , the pull-out bending moment M_{PO} can be determined only numerically from eqn(19), since $r_T(\xi)$ is a nonlinear function of M_{PO} .

Further, a collapse mechanism to be considered for a R.C. beam cross-section is the unstable fracture of concrete, which occurs when a crack instantaneously grows provoking the beam failure. The fracture bending moment M_{Fr} can be computed for a cracked T cross-section by equalling the maximum value of the stress-intensity factor (eqn(8)) to the concrete fracture toughness K_{IC} . The global internal reactions M^* and F^* to be inserted into eqn(8) are given by eqns(17) and (18), where $M = M_{Fr}$ and, if $M_{Fr} > M_{P,PO}$ (with $M_{P,PO} = \min(M_P, M_{PO})$), the force F has to be assumed equal to either F_P in the case of steel plastic flow or F_{PO} in the case of pull-out of the longitudinal bars. Note that, analogous to the previous cases of M_P and M_{PO} , the fracture bending moment M_{Fr} can be computed only numerically, since $R_F(\xi)$ and $R_M(\xi)$ are nonlinear functions of M_{Fr} .

4 R.C. BEAM UNDER CYCLIC BENDING

Now consider a cracked R.C. beam cross-section under unidirectional cyclic bending moment (varying between 0 and the maximum value M, Fig.3) and, *in the first stage of the analysis, assume that the crack does not propagate*.

If the force acting on the longitudinal steel reinforcement in correspondence to the maximum value M of the cyclic bending moment (point L_1 in Fig.3) is equal to $F_{P,PO}$, with $F_{P,PO} = \min(F_P, F_{PO})$, a residual rotation of the cross-section remains after unloading (point U_1 in Fig.3) due to either the steel plastic flow (if $F_P < F_{PO}$) or the reinforcement slippage (if $F_{PO} < F_P$). Consequently, during the unloading process (from L_1 to U_1), concrete tends to go back to its initial position and, after unloading (point U_1), compresses the steel reinforcement with an unknown force F (which is a tensile force on the concrete cross-section). The above residual rotation after unloading is determined by inserting the following values of global internal reactions into eqn(12):

$$M^* = F \ (b/2 - h) \tag{23}$$

$$F^* = F \tag{24}$$

with $$e_U = M^* / F^* = (b/2 - h) \quad . \tag{25}$$

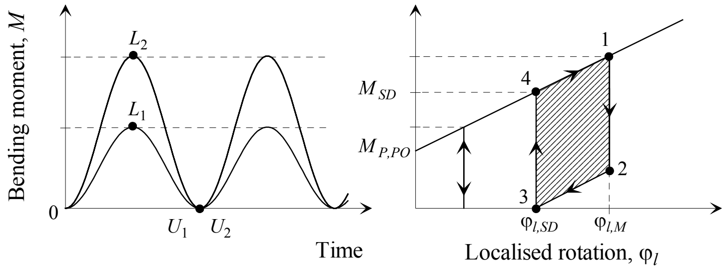

Figure 3: Elastic shake-down ($M_{P,PO} < M < M_{SD}$) and plastic shake-down ($M_{SD} < M < M_{Fr}$).

On the other hand, the global internal reactions in correspondence to the maximum value M of the cyclic bending moment (point L_1 in Fig.3) are expressed by:

$$M^* = M - F_{P,PO}\,(b/2-h) \tag{26}$$

$$F^* = -F_{P,PO} \tag{27}$$

The eccentricity e_L is given by the ratio between such expressions of internal reactions, whereas the under-loading rotation is computed by inserting the above values of M^* and F^* into eqn(12).

By assuming a rigid-perfectly plastic behaviour for the steel reinforcement, the residual rotation is equal to the under-loading rotation (see elastic shake-down in Fig.3, on the right-hand side) [5]. Such a compatibility condition allows us to obtain the relationship between the maximum value M of the applied cyclic bending moment and the steel compression force F. From such a relationship, the bending moment M_{SD} of plastic shake-down, i.e. the lowest value of the maximum bending moment M for which the steel compression force F is equal to $F_{P,PO}$, can be obtained :

$$1 = \left(\frac{M_{SD}}{M_{P,PO}} - 1\right) \frac{\lambda_{\ell,M,T}(e_L)}{\lambda_{\ell,M,T}(e_U)} \frac{\left[\frac{1}{2} - \frac{h}{b} + r_T\left(\xi, e_L\right)\right]}{\left[\frac{1}{2} - \frac{h}{b} + r_T\left(\xi, e_U\right)\right]} \tag{28}$$

where $\lambda_{\ell,M,T}$ and r_T (eqn(15) and eqn(20), respectively) depend on the above eccentricities e_L and e_U. Analogous to the previous cases of M_P, M_{PO} and M_{Fr}, M_{SD} can be computed only numerically from eqn (28), since $\lambda_{\ell,M,T}$ and r_T are nonlinear functions of M_{SD}.

If the maximum value M of cyclic loading (point L_2 in Fig.3) is greater than M_{SD}, the energy dissipated in each cycle (dashed area 1-2-3-4) can be obtained as follows:

$$\frac{work}{cycle} = M_{SD}\,\left(\varphi_{\ell,M} - \varphi_{\ell,SD}\right) \tag{29}$$

where $\varphi_{\ell,M}$ is determined by substituting the global internal reactions (eqns (26) and (27)) into eqn(12). Analogously, $\varphi_{\ell,SD}$ is computed by inserting eqns (26) and (27), with $M = M_{SD}$, into eqn(12). Note that the result of eqn(29) is slightly approximate since the segments 4-1 and 2-3 in Fig.3 are not exactly linear and, consequently, the dashed area is not exactly a parallelogram.

If the crack propagates in a stable manner [6,7], it is useful to consider all the bending moment expressions in a dimensionless form, dividing them by $K_{IC}\, b^{3/2}\, t$. The dimensionless bending moments $\tilde{M}_{SD}$ and $\tilde{M}_{Fr}$ against the relative crack depth ξ represent two curves which intersect at a critical point with coordinates (ξ_{CR}, $\tilde{M}_{CR}$) (Fig.4a). Note that, by increasing ξ from the initial

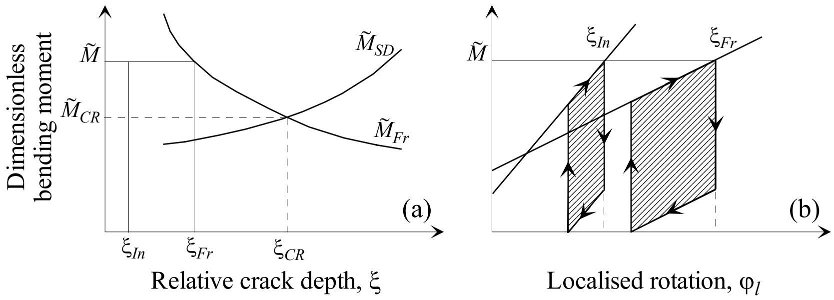

Figure 4: Plastic shake-down with hysteretic loops from ξ_{In} to ξ_{Fr}.

value ξ_{In} to the unstable fracture value ξ_{Fr} , the slope of the hardening line in a diagram of bending moment against cross-section rotation (Fig.4b) decreases monotonically and significantly, while the value of $\tilde{M}_{P,PO}$ increases very slightly.

If the dimensionless maximum bending moment $\tilde{M}$ is greater than or equal to $\tilde{M}_{CR}$ (Fig.4), fatigue crack growth occurs from ξ_{In} to ξ_{Fr} , and a hysteretic loop is described for each loading cycle. Finally, fatigue life can be determined by numerically integrating the Paris law from ξ_{In} to ξ_{Fr}.

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