A GLOBAL ENERGY CRITERION FOR CRACK PROPAGATION USING THE EXTENDED FINITE ELEMENT METHOD

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ABSTRACT

The numerical analysis of brittle materials is characterized by softening behavior associated with strain localization in the post-cracking regime. In the Extended Finite Element Method (X-FEM), which is used in this paper, discontinuities are introduced into the finite element approximation using the Partition of Unity property of the standard finite element shape functions. The analysis of crack propagation using the X-FEM crucially depends on the crack growth criterion [4], which determines whether a crack is going to propagate and if so in which direction. In this paper an energy based criterion for the prediction of the propagation direction is introduced. The basic idea of this criterion is to calculate the propagation direction by maximizing the global energy release rate of the respective structure. Therefore an additional global degree of freedom φ is introduced which is associated with the angle of the crack tip segment. To this end, a C3-continuous crack tip function is proposed for the approximation of the displacement field in the vicinity of the crack tip. The crack propagation direction is then obtained by the solution of the global coupled equation system defined by the stationary point of the total potential Π_{tot}. To illustrate the performance of the method a simple numerical example is presented. So far the proposed crack growth criterion is restricted to linear elastic fracture. In contrast to crack growth criteria based on Linear Elastic Fracture Mechanics the proposed criterion is also suited for the modeling of cohesive cracks.

1 INTRODUCTION

For the numerical analysis of structures made of brittle materials such as concrete adequate methods for the numerical representation of cracks are required. One of the most promising approaches is the representation of cracks as surfaces of discontinuous displacements within the respective finite elements. In the Extended Finite Element Method (X-FEM), which is used in this paper, discontinuities are introduced into the finite element approximation using the Partition of Unity property of the standard finite element shape functions. Therefore, and in contrast to other discrete crack models, cracks are not limited to element boundaries but can be located arbitrarily in the finite element mesh. The method was first proposed by Moës, Dolbow & Belytschko [1] and has been extended recently for the modeling of cohesive cracks [2, 3].

The analysis of crack propagation using discrete crack models crucially depends on the crack growth criterion. After crack initiation, at each time step it has to be checked whether a crack is going to propagate or not. If crack growth is signalled it has to be determined in which direction the crack will propagate. An incorrect prediction of the crack growth direction leads to locking effects and therefore to unreasonable results. This phenomenon has been reported previously in [4]. It was shown that crack growth criteria based on the maximum principle stress direction do not seem to be generally applicable. In particular for Mode I fracture biaxial tensile stress states around the crack tip may occur, which lead to a wrong prediction of the crack path. This situation may be improved by increasing the resolution of the stress field in the vicinity of the crack tip. However, the problem of severe mispredictions of the crack path cannot be completely eliminated. In the context of Linear Elastic Fracture Mechanics various crack growth criteria exist which, according to the experience of the authors, yield excellent results. Nevertheless, a consistent transformation of these criteria to the modeling of cohesive cracks is difficult. Hence, in this paper an energy based criterion for the prediction of the propagation direction of cracks in the context of the X-FEM is proposed. Although the applications of the criterion presented in this paper are restricted to linear elastic fracture problems, its extension to the modeling of cohesive cracks is straightforward.

The main idea of this criterion is to calculate the propagation direction by maximizing the global energy release rate of the respective structure. This condition is fulfilled by the propagation angle that leads to a stationary point of the total potential Π_{tot}. In the presented formulation an additional global degree of freedom φ is introduced, which is associated with the angle of the crack tip segment. The angle φ is then computed from a global coupled system of equations using the necessary and sufficient condition for the stationarity of Π_{tot}

$$\delta\,\Pi_{tot}(\boldsymbol{u}^e, \varphi) = \frac{\partial \Pi_{tot}}{\partial \boldsymbol{u}^e} + \frac{\partial \Pi_{tot}}{\partial \varphi} = 0, \tag{1}$$

where $\boldsymbol{u}^e$ consists of the regular and the enhanced degrees of freedom.

Section 2 contains an overview of the used Extended Finite Element Method. In Section 3 the algorithmic formulation of the energy based crack propagation modeling is described. A numerical example is presented in Section 4 to demonstrate the performance of the method.

2 EXTENDED FINITE ELEMENT MODEL

In this section a concise description of the used Extended Finite Element Method is given. The description is focused on the aspects relevant for the formulation of the energy based crack propagation criterion which is presented in Section 3. For a more detailed and complete description of the Extended Finite Element Method we refer to [1, 5].

2.1 Kinematics

Consider a body $\mathcal{B}$ whose domain Ω is separated into two parts Ω^+ and Ω^- by means of a localization surface $\partial_S \Omega$. The displacement field $\boldsymbol{u}$ of this body can be decomposed into a continuous part $\bar{\boldsymbol{u}}$ and a discontinuous part $\check{\boldsymbol{u}}$

$$\boldsymbol{u}(\boldsymbol{x}) = \bar{\boldsymbol{u}}(\boldsymbol{x}) + \check{\boldsymbol{u}}(\boldsymbol{x}), \quad \forall\, \boldsymbol{x} \in \Omega, \quad \text{with} \quad \check{\boldsymbol{u}}(\boldsymbol{x}) = S_S(\boldsymbol{x})\, \hat{\boldsymbol{u}}(\boldsymbol{x}), \tag{2}$$

where $\bar{\boldsymbol{u}}$ and $\hat{\boldsymbol{u}}$ are continuous functions in the domain Ω and S_S is the Sign function

$$S_S(\boldsymbol{x}) = \left\{ \begin{array}{rl} 1 & \forall \boldsymbol{x} \in \Omega^+ \\ -1 & \forall \boldsymbol{x} \in \Omega^- \end{array} \right. . \tag{3}$$

In the proposed model the approximation of the function $\hat{\boldsymbol{u}}$ is enhanced by the continuous part of the crack tip function as shown in Subsection 2.2. The Sign function S_S can be expressed in terms of the Heaviside function H_S centered on the localization surface $\partial_S \Omega$

$$S_S(\boldsymbol{x}) = 2\, H_S(\boldsymbol{x}) - 1. \tag{4}$$

The geometrically linear strain field $\boldsymbol{\varepsilon}$ is obtained by taking the symmetric gradient of $\boldsymbol{u}$ according to Equation (2)

$$\boldsymbol{\varepsilon}(\boldsymbol{u}) = \nabla^S \boldsymbol{u} = \underbrace{\nabla^S \bar{\boldsymbol{u}} + S_S \, \nabla^S \hat{\boldsymbol{u}}}_{regular} + \underbrace{2\,(\hat{\boldsymbol{u}} \otimes \boldsymbol{n})^S \delta_S}_{singular}. \tag{5}$$

The singular part of Equation (5) is expressed in terms of the Dirac-delta distribution using the classical result for the gradient of the Heaviside function $\nabla H_S = \boldsymbol{n}\,\delta_S$, where $\boldsymbol{n}$ is the unit normal vector on the surface of discontinuity. Since the presented formulation is restricted to linear elastic fracture the singular part of the geometrically linear strain tensor can be neglected in the following.

2.2 Enhanced displacement approximation

In the X-FEM the Partition of Unity Method [6] is used to locally enhance the displacement approximation where a crack has opened. In the presented formulation not only the Sign function but also a crack tip function G_S shown in Figure 1b is proposed to enhance the approximation of the displacement field in the cracked solids:

$$G_S = \begin{cases} \sin(\frac{\theta}{2})\left(\frac{35}{l^4}\,r^4 - \frac{84}{l^5}\,r^5 + \frac{70}{l^6}\,r^6 - \frac{20}{l^7}\,r^7\right) & \forall r \leq l \\ \sin(\frac{\theta}{2}) & \forall r > l \end{cases} \tag{6}$$

θ and r are the local polar coordinates at the crack tip and l is a length scale typically equal to the length of a crack segment. Using a crack tip enhancement function, crack tips do not have to be located on element boundaries, but can be located arbitrarily in the finite element mesh. Except for the jump across the localization surface, the enhancement function G_S is C3 continuous in the domain Ω. Therefore the spatial derivatives of G_S do not contain a singularity at the crack tip. This enables a consistent linearization of the Euler-Lagrange equations(see Section 3).
As mentioned above, the function G_S is discontinuous across the localization surface $\partial_S\Omega$. Similar to the discontinuous part of the displacement field the function G_S is decomposed into a product of the Sign function S_S and the continuous function G_S^*

$$G_S = S_S\,|G_S| = S_S\,G_S^*. \tag{7}$$

The reason for this decomposition is to provide a consistent notation in the following section. Using standard finite element shape functions as a partition of unity, a finite element approximation which includes all three parts, the standard finite element approximation, the Sign function S_S and the crack tip enhancement function G_S can be written as

$$\boldsymbol{u} = \underbrace{\sum_{i=1}^{n_r} N_i\,u_i^{er}}_{\bar{\boldsymbol{u}}} + S_S(\varphi)\underbrace{\left(\sum_{i=1}^{n_c} N_i\,u_i^{ec} + G_S^*(\varphi)\sum_{i=1}^{n_t} N_i\,u_i^{et}\right)}_{\hat{\boldsymbol{u}}}, \tag{8}$$

where u_i^{er} are the regular degrees of freedom, u_i^{ec} are the enhanced degrees of freedom associated with the Sign function S_S, u_i^{ec} are the enhanced degrees of freedom associated with the crack tip enhancement function G_S and φ is the degree of freedom associated with the angle of the crack tip segment. The formulation of S_S and G_S^* with respect to φ will be

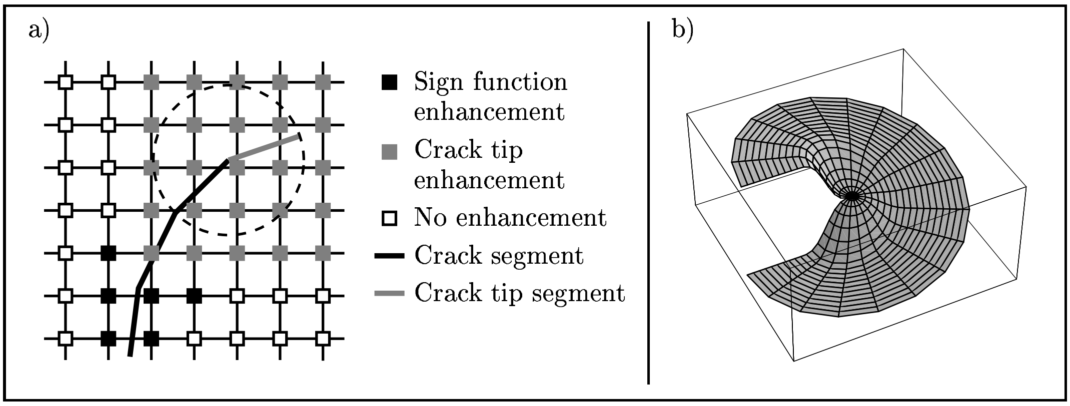

Figure 1: Nodal enhancement: a) enrichment strategy at the crack tip; b) crack-tip enhancement function G_S

explained in Section 3. As shown in Figure 1a all elements which are located in the circular area around the root of the crack tip segment are completely enhanced by the crack tip enhancement function G_S. The radius of the circle should not be less than the length of the crack tip segment to avoid a change of the enhancement during an iteration. All nodes whose support is cut by a crack and are not already enhanced by the crack tip function are enhanced by the Sign function.

3 ENERGY BASED MODELING OF CRACK PROPAGATION

The basic idea of energy based crack propagation modeling is to calculate the propagation direction by maximizing the global energy release rate of the structure. In this Section the main ingredients of this formulation are summarized. In particular, the introduction of the additional global degree of freedom φ associated with the direction of the crack tip segment and the consistent linearization of the field equations are described.

3.1 Introduction of the global degree of freedom φ

To account for the global degree of freedom φ the enhancement functions S_S and G_S have to be formulated in terms of φ. The Sign function S_S at the crack tip segment is defined as

$$S_S(\boldsymbol{x}, \varphi) = \text{sign}(d(\boldsymbol{x}, \varphi)) \qquad \text{with} \qquad d(\boldsymbol{x}, \varphi) = \boldsymbol{n}(\varphi)\,(\boldsymbol{x} - \boldsymbol{x_{cr}}), \tag{9}$$

where $\boldsymbol{x_{cr}}$ are the coordinates of the root of the crack tip segment which are invariant with respect to φ. Similar to the Sign function the local polar coordinates at the crack tip r, θ can be formulated in terms of φ and and $\boldsymbol{x}$ which leads to $G_S(\boldsymbol{x}, \varphi)$. As mentioned above the chosen crack tip enhancement function G_S is C3 continuous in the domain Ω and thus the derivatives of G_S are well defined. The first and second derivative of S_S with respect to φ lead to

$$\frac{\partial S_S}{\partial \varphi} = 2\,\frac{\partial d}{\partial \varphi}\,\delta_S \qquad \text{and} \qquad \frac{\partial^2 S_S}{\partial^2 \varphi} = 2\,\frac{\partial d}{\partial \varphi}\,\frac{\partial d}{\partial \varphi}\,\delta_S' + 2\,\frac{\partial^2 d}{\partial^2 \varphi}\,\delta_S, \tag{10}$$

where δ_S is the Dirac-delta distribution centered on the localization surface.

3.2 Finite Element Formulation

Inserting the geometrically linear strain tensor (5) into the principle of the minimum of the total potential, neglecting body forces, leads to

$$\Pi_{tot} = \Pi_{int} + \Pi_{ext} = \text{stat.} \qquad \text{with} \qquad \Pi_{ext} = -\int_{\Gamma_\sigma} \hat{\boldsymbol{u}} \cdot \boldsymbol{t}^* \, \mathrm{d}\Gamma \tag{11}$$

$$\Pi_{int} = \int_{\Omega} (\boldsymbol{\nabla}^S \bar{\boldsymbol{u}} : \mathbb{C} : \boldsymbol{\nabla}^S \bar{\boldsymbol{u}} + \boldsymbol{\nabla}^S \hat{\boldsymbol{u}} : \mathbb{C} : \boldsymbol{\nabla}^S \hat{\boldsymbol{u}} + 2\, S_S\, \boldsymbol{\nabla}^S \bar{\boldsymbol{u}} : \mathbb{C} : \boldsymbol{\nabla}^S \hat{\boldsymbol{u}}) \, \mathrm{d}V$$

As mentioned before the formulation is so far restricted to Linear Elastic Fracture Mechanics. Since for linear elastic fracture stress free crack faces are assumed, the singular part of the strain tensor can be neglected. Nevertheless a consideration of the singular part is straightforward. For reasons of simplicity it is assumed that the enhanced part of the displacement field is zero where boundary conditions are imposed. Therefore the enhanced part of the displacement field does not affect Π_{ext}. Taking the first and second variation of Π_{int} and considering the derivatives of S_S given in Equation (10) together with the integration rules for the Dirac-delta distribution and its derivative leads to the coupled stiffness matrix and the internal load vector obtained in the general format

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}^{\bar{\boldsymbol{u}},\bar{\boldsymbol{u}}} & \boldsymbol{K}^{\bar{\boldsymbol{u}},\hat{\boldsymbol{u}}} & \boldsymbol{K}^{\bar{\boldsymbol{u}},\varphi} \\ \text{Sym.} & \boldsymbol{K}^{\hat{\boldsymbol{u}},\hat{\boldsymbol{u}}} & \boldsymbol{K}^{\hat{\boldsymbol{u}},\varphi} \\ \text{Sym.} & \text{Sym.} & \boldsymbol{K}^{\varphi,\varphi} \end{bmatrix}, \quad \boldsymbol{f}_{int} = \begin{bmatrix} \boldsymbol{f}^{\bar{\boldsymbol{u}}} \\ \boldsymbol{f}^{\hat{\boldsymbol{u}}} \\ \boldsymbol{f}^{\varphi} \end{bmatrix}. \tag{12}$$

4 NUMERICAL EXAMPLE

The numerical example investigated in this section consists of a square shaped panel as shown in Figure 2. An initial vertical crack with a length of $L/5$ and a crack tip segment with the length $L/10$ are assumed. Two different loading conditions are investigated, the first loading condition is a pure tensile load ($u_v = 0$), the second is a mixed mode loading condition ($u_v = u_h$). Both loading conditions are investigated using two different finite element meshes, a regular mesh consisting of 81 four-node finite elements and an unstructured mesh consisting of 64 four node finite elements. For the pure tension test the initial propagation angle φ was set to $-10°$ and for the mixed mode test it was set to $-30°$.
Figure 3 illustrates the convergence behavior for both finite element discretizations. In all

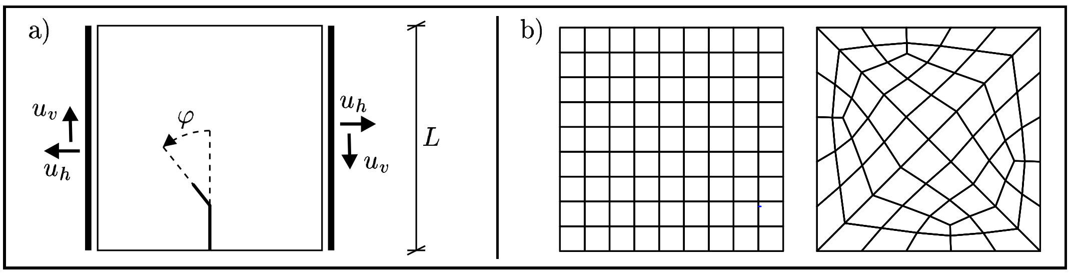

Figure 2: Square shaped slab: a) loading conditions, initial crack and definition of φ; b) regular and unregular finite element mesh

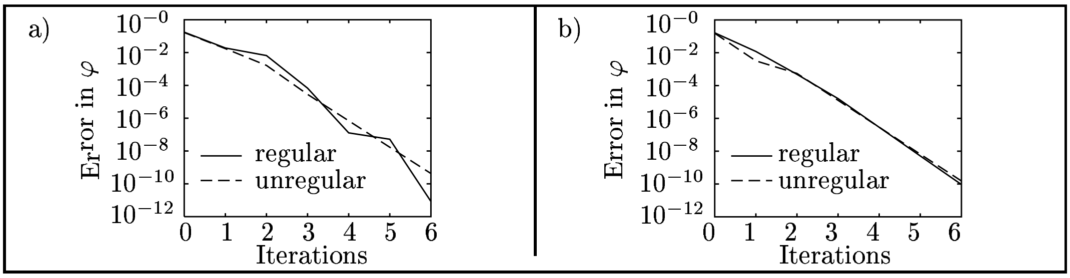

Figure 3: Square shaped slab: a) rate of convergence for mode-I loading conditions; b) rate of convergence for mixed mode loading conditions

cases an (almost) quadratic rate of convergence was obtained. The calculated propagation angle for the pure tension test finally was obtained as $\varphi = 0°$ for the regular mesh and as $\varphi = 0.5°$ for the unregular mesh. For the mixed mode test a propagation angle of $\varphi = -21°$ for the regular and $\varphi = -21.6°$ for the unregular mesh was computed.

5 CONCLUSION

In this paper an energy based criterion for the prediction of the propagation cracks in quasi-brittle materials was presented in the context of the Extended Finite Element Method. In the proposed formulation the propagation angle is considered as an additional global degree of freedom which is calculated by solving the global coupled system of equations associated with the maximum energy release rate. This method was successfully applied to the propagation of non-cohesive cracks under mode-I and mixed mode loading conditions using structured and unstructrured meshes. However, the proposed criterion holds for cohesive as well as for non-cohesive cracks. Hence, future work will be concerned with the extension of the method to cohesive cracks and with modifications to increase the radius of convergence using a stabilized NEWTON algorithm.

6 REFERENCES

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