AN ‘EXTENDED’ ANISOTROPIC DAMAGE MODEL BASED ON YOUNG/POISSON DECOMPOSITION

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ABSTRACT

One of the open challenges in anisotropic damage modeling is formulating damaged elasticity approaching general orthotropy. In this paper, with reference to previous work by the same authors, an ‘extended’ constitutive formulation of isotropic and anisotropic damage is proposed, which is based on a Young’s modulus/Poisson’s ratio decomposition of the initial isotropic compliance. Orthotropic damage is described through a second-order tensor damage variable. Main focus is on the secant relations and on the dependence of the engineering elastic parameters with damage as a function of a scalar characteristic parameter defining the slope of linear constrained damage paths in the plane of logarithmic damage variables.

1 INTRODUCTION

Anisotropic damage modeling still poses a number of challenges. One of them is formulating secant moduli approaching general orthotropic representations, while preserving at the same time a convenient modular structure and manageable number of damage variables. The constitutive formulations can be developed within the well-established framework of Continuum Damage Mechanics (CDM), which allows to employ convenient concepts (though not mandatory) such as effective stresses and strains, damage-effect tensors, strain or energy equivalence, and so on. The developments on CDM have given rise to a considerable literature. For the sake of conciseness, reference is made here essentially to previous work by the same authors [1,2], papers which contain as well extensive reference lists on the topic.

In the recent past, the authors have developed a ‘basic’ model of anisotropic damage characterized by secant moduli endowed with simple expressions, which corresponds to a restricted 5-coefficient form of orthotropic degradation and fits well into the CDM framework through the assumption of energy equivalence and the adoption of a second-order tensor damage variable [1]. Then, the authors have undertaken the effort of generalizing the representation of anisotropic damage, with the long-term objective of eventually approaching 9-coefficient general orthotropy. The first step is to increase from 5 to 6-coefficient orthotropy. This is done by combining the ‘basic’ formulation with a 2-coefficient isotropic degradation, which can be developed in various ways, depending on the isotropic stiffness or compliance that is taken as reference for the virgin material. An ‘extended’ formulation has been developed along that line, which is based on a volumetric/deviatoric decomposition of the initial stiffness and compliance [2]. The formulation considers both secant and tangent moduli and seems to be suited best for materials displaying prevailing deviatoric damage.

In this paper, an additional ‘extended’ formulation better suited for concrete and other quasi-brittle materials is proposed, which is based on a Young’s Modulus/Poisson’s ratio decomposition of the initial isotropic compliance. Main focus is on the secant relations and on the dependence of the engineering elastic parameters with damage as a function of a scalar characteristic model parameter defining the slope of 'single-dissipative' linear constrained damage paths in the plane of logarithmic damage variables. In the isotropic case all engineering moduli and Poisson's ratio experience smooth decreasing trends at increasing damage.

Later, the 'extended' formulation should be made compatible with evolution laws in 'pseudo-logarithmic' space of damage, as done for the previous K/G-'extended' formulation, and analytical solutions for simple loading cases should be derived same as done for the 'basic' formulation (e.g. uniaxial tension, pure shear), before final implementation in constitutive drivers would be considered to evaluate efficiency, correctness and consistency, especially with respect to more involved loading scenarios involving rotation of the principal directions of stress/strain and damage (as done in [1]).

Notation. Second-order tensors are identified by boldface characters, whereas fourth-order tensors are denoted by blackboard-bold fonts (e.g. $\mathbb{C}$). Symbol ':' denotes the inner product with double contraction. The dyadic product is indicated with '$\otimes$', whereas '$\overline{\underline{\otimes}}$' denotes the symmetrized outer product; componentwise: $(\mathbf{A}\otimes\mathbf{B})_{ijkl}=A_{ij}B_{kl}$ and $(\mathbf{A}\,\overline{\underline{\otimes}}\,\mathbf{B})_{ijkl}=(A_{ik}B_{jl}+A_{il}B_{jk})/2$. $\mathbf{I}$ and $\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I}$ are respectively the second-order and symmetric fourth-order identity tensors.

2 SECANT RELATION AND ISOTROPIC REFERENCE COMPLIANCE

Considering pure elastic degradation (no irreversible strains at unloading), it is assumed that the constitutive relation is characterized by a linear secant law. At any damage state the (small) strain tensor $\boldsymbol{\epsilon}$ and stress tensor $\boldsymbol{\sigma}$ are related by:

$$\boldsymbol{\epsilon} = \mathbb{C}(\mathbb{C}_0, \boldsymbol{\mathcal{D}}) : \boldsymbol{\sigma} \ , \tag{1}$$

where $\mathbb{C}$ is the positive-definite fourth-order compliance tensor, endowed with major and minor symmetries. In the initial (virgin) state the material is characterized by undamaged compliance $\mathbb{C}_0$. In Eq. (1) it is also assumed that the compliance tensor $\mathbb{C}$ is a function of a generally-defined damage variable $\boldsymbol{\mathcal{D}}$, which may be scalar, vector- or tensor-valued (here scalar or symmetric second-order tensor variables are assumed). Obviously, the damaged compliance also depends explicitly on its initial value $\mathbb{C}_0$ at virgin state. Such reference is assumed here to be isotropic.

To develop a damage model of initially-isotropic materials, different forms of the reference isotropic compliance could be considered. In the 'extended' model [2], reference was made to the bulk and shear moduli and relevant compliance representation in terms of volumetric and deviatoric projection operators. This allowed a number of convenient features, especially the possibility to use the algebraic properties of the idempotent projection operators [6]. However, the final trends experienced by the elastic parameters were not entirely satisfactory for concrete or rock-like materials. For example, Poisson's ratio increased with prevailing deviatoric damage, while decreased to negative values for prevailing volumetric damage. On the other hand, one may expect a decreasing, positive Poisson's ratio for concrete. Other authors have started from different isotropic references, for instance by referring to the elongation and bulk moduli [3,4], which is claimed sufficient to describe completely the orientation distribution functions of the elastic properties. This calls for a general investigation on the more appropriate reference states apt to address damage for specific materials.

In the present model we refer to the following form of the undamaged isotropic compliance:

$$\mathbb{C}_0 = \frac{1+\nu_0}{E_0}\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} - \frac{\nu_0}{E_0}\,\mathbf{I}\otimes\mathbf{I} \quad \Rightarrow \quad \mathbb{C}_0 = \frac{1}{E_0}\,\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} + \frac{\nu_0}{E_0}\left(\mathbf{I}\,\overline{\underline{\otimes}}\,\mathbf{I} - \mathbf{I}\otimes\mathbf{I}\right) , \tag{2}$$

in terms of undamaged Young's modulus E_0 and Poisson's ratio ν_0. Here the two fourth-order tensor terms have the following classical 6×6 matrix representations in Kelvin notation (see e.g. [6]):

$$[\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} ; \qquad [\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \mathbf{I}\otimes\mathbf{I}] = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} . \tag{3}$$

3 ISOTROPIC DAMAGE

Considering first isotropic damage based on the reference isotropic compliance (2), the damaged (increased) compliance at the current isotropic damage state reads:

$$\mathbb{C} = \frac{1}{E}\,\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} + \frac{\nu}{E}\left(\mathbf{I}\,\underline{\overline{\otimes}}\,\mathbf{I} - \mathbf{I}\otimes\mathbf{I}\right) , \tag{4}$$

where E and ν are current Young's modulus and Poisson's ratio.

Different damage variables may be introduced at this stage to describe the evolution of the secant parameters. We do it according to the following assumptions (within the spirit of the so-called classical (1$-D$)-isotropic damage model):

$$\frac{1}{E} = \frac{1}{1-D_E}\,\frac{1}{E_0} ; \qquad \frac{\nu}{E} = \frac{1}{1-D_{E/\nu}}\,\frac{\nu_0}{E_0} . \tag{5}$$

Scalar damage variables $0 \leq D_E < 1$ and $0 \leq D_{E/\nu} < 1$ model the increase of the compliance terms related to $1/E$ and ν/E. Notice that the model is 'bi-dissipative' at this stage if the two damage variables evolve independently. To work out the direct dependence of ν with damage, one gets: $\nu/\nu_0 = (1-D_E)/(1-D_{E/\nu}) = 1-D_\nu$, or $D_\nu = (D_E - D_{E/\nu})/(1-D_{E/\nu})$. Then, if a decrease of the Poisson's ratio has to be prescribed ($\nu/\nu_0 \leq 1$), evolution laws of D_E and $D_{E/\nu}$ should be defined such that the constraint $D_{E/\nu} \leq D_E$ would hold at each damage state. If $D_{E/\nu} = D_E$, Poisson's ratio becomes constant, which originates the classical (1$-D$)-isotropic damage model referred-to above.

Alternative scalar damage variables can be introduced in place of D_E and $D_{E/\nu}$, specifically the integrity variables $\bar{\phi}_E$, $\bar{\phi}_{E/\nu}$, with complementary variation between 1 and 0, and the logarithmic damage variables L_E, $L_{E/\nu}$, with unbounded increase from 0 to ∞ [1,2]:

$$\begin{aligned} 1 - D_E &= \bar{\phi}_E^{\,2} = e^{-L_E} ; & \bar{\phi}_E &= \sqrt{1-D_E} , & L_E &= \ln\frac{1}{1-D_E} ; \\ 1 - D_{E/\nu} &= \bar{\phi}_{E/\nu}^{\,2} = e^{-L_{E/\nu}} ; & \bar{\phi}_{E/\nu} &= \sqrt{1-D_{E/\nu}} , & L_{E/\nu} &= \ln\frac{1}{1-D_{E/\nu}} . \end{aligned} \tag{6}$$

To keep the model 'single-dissipative', the following links between the logarithmic damage variables L_E and $L_{E/\nu}$ and a single logarithmic damage variable L are prescribed:

$$L_E = (1+\eta)\,L ; \qquad L_{E/\nu} = (1-\eta)\,L . \tag{7}$$

Here, constant parameter η, with general range $-1 \leq \eta \leq 1$, fixes the slope of constrained linear paths in the plane of the logarithmic damage variables (Fig. 1), while L may be alternatively represented by damage variables D and $\bar{\phi}$ through similar links as the ones above: $L = \ln 1/(1-D) = -2\ln\bar{\phi}$.

With such assumption one obtains the following variations of E and ν with $\bar{\phi}$:

$$\frac{E}{E_0} = \bar{\phi}^{\,2(1+\eta)} ; \qquad \frac{\nu}{\nu_0} = \bar{\phi}^{\,4\eta} . \tag{8}$$

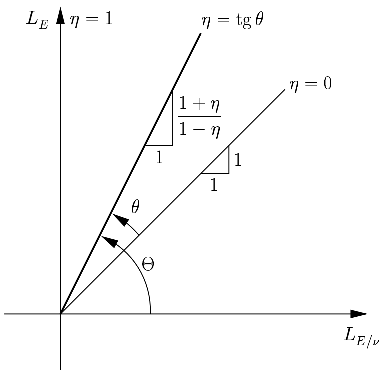

Figure 1: Single-dissipative straight paths in the plane of logarithmic damage variables $L_{E/\nu}$, L_E. The path slope in the upper half of the I quadrant is defined by constant parameter $0\leq\eta\leq 1$.

To avoid (unbounded) growth of ν, according to the constraint $D_{E/\nu}\leq D_E$, we shall restrict the range of constant η to positive values: $0\leq\eta\leq 1$. If η=0 a classical $(1-D)$-model is obtained and ν=ν_0. Bulk and shear moduli can also be derived as follows:

$$\frac{K}{K_0} = \bar{\phi}^{\,2(1+\eta)}\,\frac{1-2\nu_0}{1-2\nu_0\bar{\phi}^{\,4\eta}}\ ; \qquad \frac{G}{G_0} = \bar{\phi}^{\,2(1+\eta)}\,\frac{1+\nu_0}{1+\nu_0\bar{\phi}^{\,4\eta}}\ . \tag{9}$$

Figure 2 depicts the variation of the material parameters with both logarithmic and standard damage variables L and D for different values of $0\leq\eta\leq 1$. A linear variation of ν with D is obtained for η=$1/2$. If η=0 all moduli E, K and G decrease linearly to 0 as in the $(1-D)$-model.

4 ANISOTROPIC DAMAGE

The secant compliance at the current anisotropic damage state is prescribed in a form generalizing that of Valanis [5] in which the identity tensors are replaced by the inverses of the integrity tensors (symmetric second-order tensors varying between **I** (no damage) and **0** (full damage), see also [1]). Operating like that on the reference compliance (2) and following steps similar to those developed in [2] one obtains:

$$\mathbb{C} = \frac{1}{E_0}\,\bar{\boldsymbol{\phi}}_E^{-1}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}_E^{-1} + \frac{\nu_0}{E_0}\left(\bar{\boldsymbol{\phi}}_{E/\nu}^{-1}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}_{E/\nu}^{-1} - \bar{\boldsymbol{\phi}}_{E/\nu}^{-1}\otimes\bar{\boldsymbol{\phi}}_{E/\nu}^{-1}\right)\ , \tag{10}$$

where $\bar{\boldsymbol{\phi}}_E$ and $\bar{\boldsymbol{\phi}}_{E/\nu}$ are integrity tensors associated to factors E and E/ν. At this stage the model would then be ‘bi-dissipative’. Same as in the isotropic case, the model is turned ‘single-dissipative’ through the following product decomposition assumption:

$$\bar{\boldsymbol{\phi}}_E = \bar{\phi}^{\eta}\,\bar{\boldsymbol{\phi}}\ ; \qquad \bar{\boldsymbol{\phi}}_{E/\nu} = \bar{\phi}^{-\eta}\,\bar{\boldsymbol{\phi}}\ , \tag{11}$$

where $\bar{\boldsymbol{\phi}}$ is a common integrity tensor describing the anisotropic damage state and scalar $\bar{\phi}$ is the 1/3 power of the determinant of $\bar{\boldsymbol{\phi}}$, $\bar{\phi}$=$(\det\bar{\boldsymbol{\phi}})^{1/3}$.

Then, the current isotropic compliance can be written in final form as:

$$\mathbb{C} = \frac{1}{\hat{E}}\,\bar{\boldsymbol{\phi}}^{\,-1}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}^{\,-1} + \frac{\hat{\nu}}{\hat{E}}\left(\bar{\boldsymbol{\phi}}^{\,-1}\,\underline{\overline{\otimes}}\,\bar{\boldsymbol{\phi}}^{\,-1} - \bar{\boldsymbol{\phi}}^{\,-1}\otimes\bar{\boldsymbol{\phi}}^{\,-1}\right)\ , \tag{12}$$

similar to Valanis’ expression but with undamaged elastic parameters replaced by modified elastic parameters with hats defined as follows:

$$\hat{E} = \bar{\phi}^{\,2\eta}\,E_0 = \bar{\phi}^{\,-2}\,E\ ; \qquad \hat{\nu} = \bar{\phi}^{\,4\eta}\,\nu_0 \equiv \nu\ . \tag{13}$$

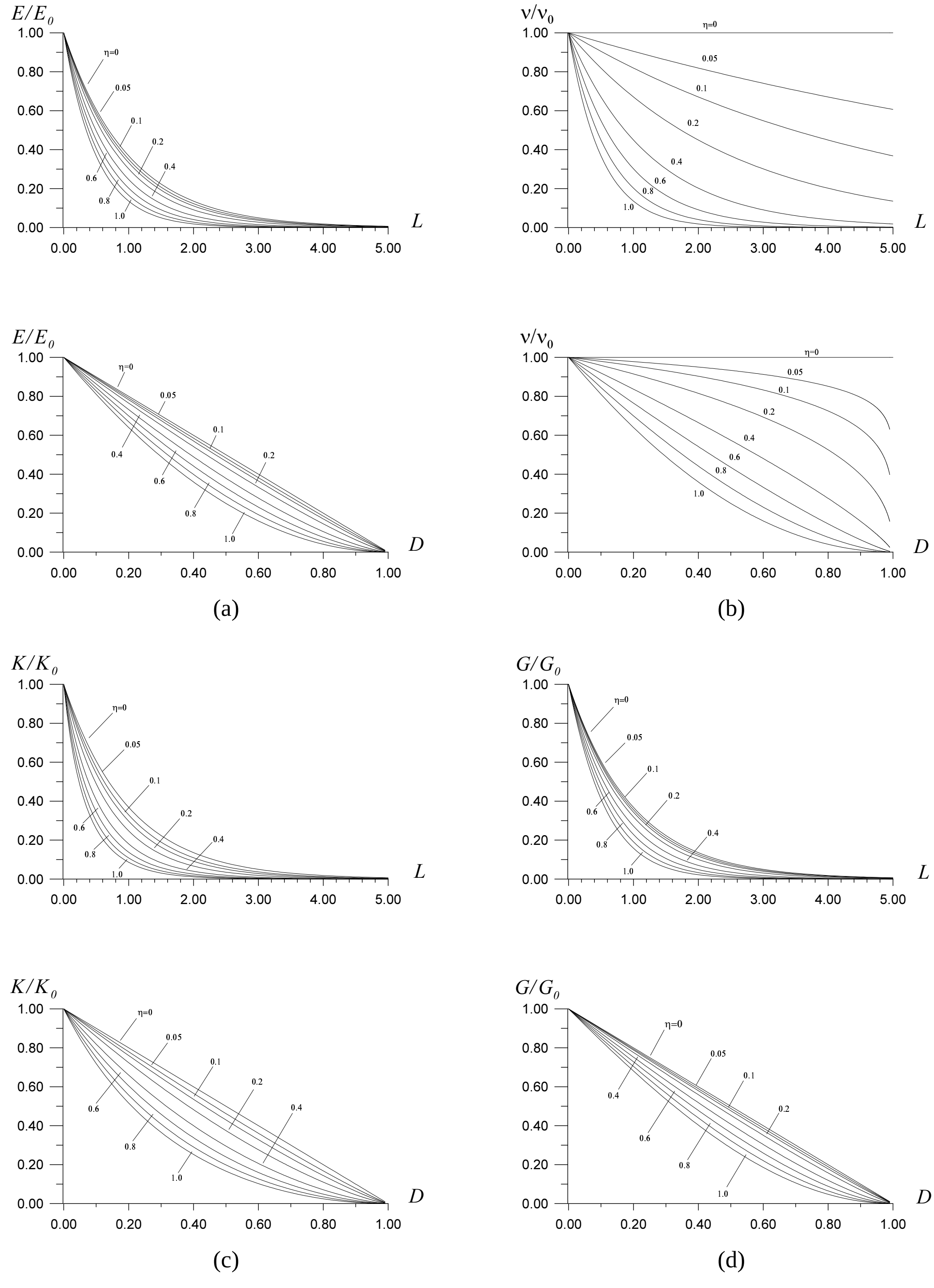

Figure 2: Elastic parameters as a function of damage variables L and D: (a) Young's modulus; (b) Poisson's ratio; (c) Bulk modulus (for ν_0=0.2); (d) Shear modulus (for ν_0=0.2).

Notice that $\hat{\nu}\equiv\nu$. Also, bulk and shear moduli with hats derived according to (13) read:

$$\hat{K}=\frac{\hat{E}}{3(1-2\hat{\nu})}=\bar{\phi}^{\,2\eta}\,\frac{1-2\nu_0}{1-2\nu_0\bar{\phi}^{\,4\eta}}\,K_0=\bar{\phi}^{\,-2}\,K;\;\hat{G}=\frac{\hat{E}}{2(1+\hat{\nu})}=\bar{\phi}^{\,2\eta}\,\frac{1+\nu_0}{1+\nu_0\bar{\phi}^{\,4\eta}}\,G_0=\bar{\phi}^{\,-2}\,G. \quad (14)$$

The orthotropic engineering elastic parameters embedded in (12) are finally obtained in the principal axes of damage as:

$$[\mathbb{C}_{\rm orth}]=\begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_2} & -\frac{\nu_{13}}{E_3} & & & \\ -\frac{\nu_{21}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{23}}{E_3} & & & \\ -\frac{\nu_{31}}{E_1} & -\frac{\nu_{32}}{E_2} & \frac{1}{E_3} & & & \\ & & & \frac{1}{2G_{12}} & & \\ & & & & \frac{1}{2G_{23}} & \\ & & & & & \frac{1}{2G_{31}} \end{bmatrix};\;\left\{\begin{array}{lll} E_I &= \bar{\phi}_I^2\,\hat{E}\,, & I=1,2,3\,; \\ G_{IJ} &= \bar{\phi}_I\bar{\phi}_J\,\hat{G}\,, & I,J=1,2\,;\,2,3\,;\,3,1\,; \\ \nu_{IJ} &= \dfrac{\bar{\phi}_J}{\bar{\phi}_I}\,\hat{\nu}\,, & I\neq J=1,2,3\,; \end{array}\right. \quad (15)$$

where the dependence on η is hidden in the parameters with hats according to (13) and $(14)_b$. The model is then characterized by 6 parameters, 3 constants (undamaged elastic constants E_0, ν_0 and path-parameter η), and 3 evolving principal values of $\bar{\boldsymbol{\phi}}$.

5 CONCLUDING REMARKS

An 'extended' formulation of anisotropic elastic damage in initially-isotropic materials has been presented. It is based on a particular decomposition of the reference isotropic compliance. A path-parameter η discriminates the damage weights of Young's modulus and Poisson's ratio. The dependence of the isotropic parameters with damage has been obtained, which shows smooth decreasing trends at increasing damage in all cases. The nine orthotropic engineering material parameters entering the secant compliance have been also characterized in terms of the damage state.

The present model is certainly viable of further developments: completion of the constitutive formulation with appropriate evolution laws and determination of the response in rates, determination of analytic solutions for simple loading paths (e.g. uniaxial tension, pure shear, etc.), implementation in a constitutive driver, exploitation of more involved loading scenarios with rotations of the principal directions of stress, strain and damage. These issues are at present still concern of investigation.

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