BOUNDARY INTEGRAL FRACTURE ANALYSIS AND HYPERSINGULAR EVALUATION

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ABSTRACT

It has recently been shown that, for mixed mode problems in two dimensions, highly accurate stress intensity factors can be obtained by combining the simple Displacement Correlation method with a new crack front element. This 'modified quarter point element' adds a cubic term to the standard quadratic element, forcing the crack opening displacement to satisfy a known constraint at the front. In the present work, this two dimensional analysis has been extended to three dimensions. The fracture computations have been carried out by solving boundary integral equations, and in particular, the hypersingular integral equation for surface traction. Part of this work has therefore involved the development of algorithms for evaluating hypersingular surface integrals. Boundary limit evaluation methods initially developed for the Galerkin formulation using linear elements have been successfully extended to higher order interpolation. These algorithms are based upon analytic integration, and directly evaluate the hypersingular integrals without reformulating or regularizing the traction equation.

1 INTRODUCTION

The key quantities of interest in computational fracture analysis are the stress intensity factors (SIFs), as these determine crack propagation. A significant advance in this area was the development of the quadratic quarter point element Henshell [1], Barsoum[2]. With this element, the square root singularity of the crack opening displacement at the tip could be easily incorporated in the numerical interpolation. Nevertheless, this method did not always result in highly accurate SIFs.

For example, consider a mixed-mode example having a pair of circular-arc cracks embedded in a plate subjected to a remote biaxial tension σ, as shown in Figure 1. The exact SIFs for this problem (as a function of the separation angle θ) are known; the dashed lines in Figure 2 are the percentage errors in the computed K_I and K_{II} obtained using the standard quarter point element. The opening mode K_I is reasonably accurate when the cracks are not interacting, but becomes steadily worse as θ approaches π. On the other hand, K_{II} is not at all reliable.

The significantly more acccurate solid lines in this figure are the results if the quadratic quarter point element is modified to include a cubic contribution Phan [3] . The purpose of this higher order term is to insure that the interpolated crack opening displacement satisfies a known constraint, the term that is linear in distance to the front must vanish Gray [4] . Further examples of the improved accuracy seen with the modified element can be found in [3].

Based upon the success of this initial work, the goal is to now extend these methods to three dimensions.

2 THREE DIMENSIONS

The first task is to establish that the constraint on the crack opening displacement (vanishing

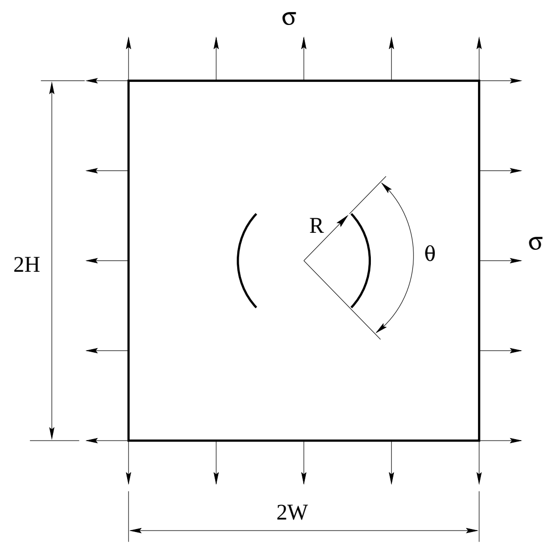

Figure 1: Pair of circular-arc cracks.

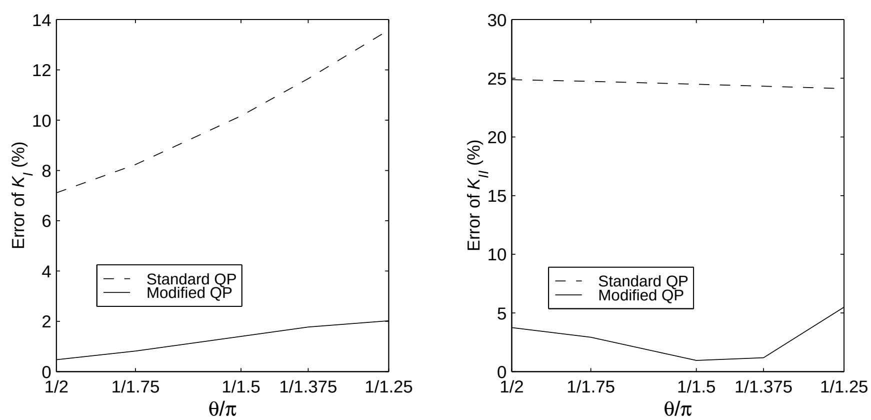

Figure 2: Effect of crack interaction on SIFs.

of the linear term) also holds in three dimensions. This has been implicitly assumed in Li [5], and has yielded very accurate SIFs. The argument is a straightforward generalization of the proof in two dimensions. If the limiting value of the traction at the crack front is computed, the linear term from the crack opening displacement yields a logarithmic singularity: as the traction must be finite, the linear term must vanish. The expression for the traction at the front is obtained from the boundary integral formulation for elasticity, discussed below.

The second major aspect of this work concerns the numerical solution of the elasticity equations. As in two dimensions, the fracture computations will be carried out using a Galerkin approximation of the corresponding boundary integral equations Bonnet [6]. In this approach, the hypersingular equation for surface traction is employed on the crack surface. This equation is obtained by diferentiating the equation for surface displacement,

$$u_k(P) + \int_{\partial\Omega} \left\{ T_{kj}(P,Q)u_j(Q) - \int_{\partial\Omega} U_{kj}(P,Q)\tau_j(Q) \right\} dQ = 0 \ , \tag{1}$$

yielding an expression for surface stress,

$$\sigma_{lk}(P) + \int_{\partial\Omega} \{S_{lkm}(P,Q)u_m(Q) - D_{lkm}(P,Q)\tau_m(Q)\} \ dQ = 0 \ . \tag{2}$$

The traction formula then results from taking the appropriate inner product. Here $u_m(Q)$ and $\tau_m(Q)$ are the components of the surface displacement and traction vectors, $\partial\Omega$ is the boundary of the domain, and $U(P,Q)$ is the well known Kelvin solution. The other kernel functions $T(P,Q)$, $D(P,Q)$, and (the hypersingular) $S(P,Q)$ are appropriate linear combinations of derivatives of U. The Galerkin form of these equations [5,7] is obtained by multiplying by a weight function and integrating with respect to P.

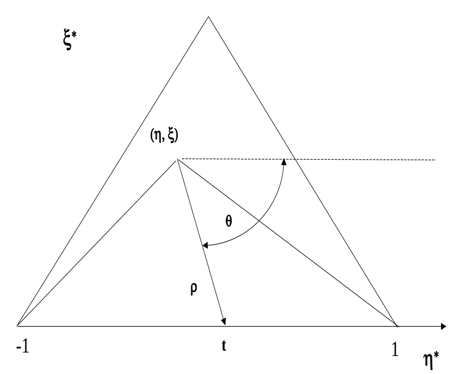

Figure 3: First polar coordinate transformation, $\{\eta^*, \xi^*\} \rightarrow \{\rho, \theta\}$, for the coincident integration.

In previous work, the evaluation of the Galerkin hypersingular integral has been successfully accomplished by applying Stokes' Theorem to regularize the integral, Li [5] and Frangi [7] . In this work, we apply a recently developed approach that evaluates the hypersingular (and singular) integrals directly, Gray [8] . This method is based upon defining the integral as a boundary limit, moving the point P off the boundary a distance ϵ, $P \rightarrow P + \epsilon\mathbf{N}$, where $\mathbf{N}$ is the unit normal at P. For $\epsilon > 0$ the integral is nonsingular, and after appropriate analytic integration, formulas for the limiting value $\epsilon \rightarrow 0$ can be derived.

For example, for the coincident integration (when the integrals with respect to P and Q are over the same element), the first step is to replace the Q parameters by polar coordinates

(ρ, θ) centered at P, as indicated in Fig. 3 For linear elements, the variable ρ can then be integrated analytically, as the distance function (which appears in the denominator of the Green's function expressions) takes the simple form

$$r^2 = \|Q - P\|^2 = a^2\rho^2 + \epsilon^2 \ . \tag{3}$$

For higher order interpolation, required for the crack tip analysis, the distance function is no longer a quadratic in ρ, and the analytic integration cannot be executed as for linear elements. A primary task of this work is to therefore extend the techniques in [8] to this situation. This is accomplished by splitting the integrand into what is essentially a 'linear interpolation' component that can be integrated analytically, and a nonsingular remainder component that can be evaluated numerically.

Not surprisingly, it is found that, as in the linear element analysis, the coincident and adjacent edge integrals are separately divergent. However, the divergent terms can be explicitly computed and shown to cancel when all integrals are summed. Thus the complete hypersingular integral is a well defined finite quantity.

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