Hamiltonian Formalisms In Elasticity - Potential Use for Fracture Mechanics

J. LI[1], N. RECHO[2]
[1]LPMTM, CNRS UPR 9001, Université Paris XIII,
99 Avenue Jean-Baptiste Clément, 93430 Villetaneuse
[2]LAMI, Université Blaise Pascal de Clermont-Ferrand,
Avenue Aristide Briand, 03100 Montluçon

ABSTRACT

The first part of this paper deals with several Hamiltonian formalisms in elasticity. We first present briefly the formalisms of Zhong and Bui, (Bui, [1]; Zhong, [2]), which resolve respectively the two-end problem and the Cauchy problem in elasticity. Then we propose a new Hamiltonian formalism, which resolves simultaneously the two problems mentioned above and it shows the link between the two formalisms. The potential use for fracture mechanics purposes is then mentioned. In fact, when traditional theories in fracture mechanics are used, asymptotic analyses are often carried out by using high-order differential equations governing the stress field near the crack tip. The solution of the high-order differential equations becomes difficult when one deals with anisotropic or multiplayer media etc. The key of our idea was to introduce the Hamiltonian system, usually studied in rational mechanics, into continuum mechanics also, one can obtain a system of first-order differential equations, instead of the high-order differential equation. This method is very efficient and quite simple to obtain solution of the governing equations of this class of problems. It allows dealing with large range of problems, which may be difficult to resolve by using traditional methods.

1 INTRODUCTION

Recently, important effort has been made in the reform of the classical theory of continuum mechanics in the frame of the Hamiltonian system. In these new approaches, the principle of Hamilton is applied in a special manner, i.e., by considering a dimensional parameter as "time". In this topic, we can distinguish two formalisms: the formalism of Bui [1] and the formalism of Zhong [2]. By seeking the variations of the couple (displacements, traction forces) on an arbitrary front in a solid when this front virtually moves from an initial position to a neighbour one, a first-order differential equation system governing the mechanical fields was explicitly established. That is the Cauchy problem in elasticity resolved by Bui [1]. On the other hand, the formalism of Zhong [2] looks like more classical. In simple words, he established an analogy between quantities in rational mechanics and those in continuum mechanics. For example, a dimensional coordinate in continuum mechanics is considered as time in rational mechanics; the displacement vector as the generalized coordinates; the strain energy density as the Lagrange function and so on. This analogy leads to the canonical equations of Hamilton governing the mechanical fields in elastic bodies. The main advantage of these approaches is that the fundamental equations can directly be resolved. The traditional semi-inverse method is then replaced by a direct, systematic and more structural resolution method.

2 ZHONG'S FORMALISM; THE TWO-END PROBLEM

Let consider a solid V described a coordinate system Z in which z is one chosen coordinate. Let consider now q the displacements in Z system associated to neighbour displacements, $q+\delta q$. One notes $\dot{\boldsymbol{q}} = \dfrac{\partial \boldsymbol{q}}{\partial z}$. If we suppose that the displacements are imposed at $z = z_0$ and $z = z_1$, named the two end points. We have then:

$$\delta \boldsymbol{q}(z = z_0) = \delta \boldsymbol{q}(z = z_1) = \boldsymbol{0} \tag{1}$$

Let write the total potential energy Π of the solid:

$$\Pi = \int_{z_0}^{z_1} \iint_S (U_0 - W) dS dz = \int_{z_0}^{z_1} L dz \quad avec \quad L = \int_S (U_0 - W) dS \tag{2}$$

Where U_0 is the strain energy density and W is the work density of the external forces. We define the Lagrange function as the integral over S. If S is the constant along z and we neglect the body forces and we just consider a volume element inside the solid, we can write $L=U_0 - W$. In general, L is a function of $\boldsymbol{q}$ and $\dot{\boldsymbol{q}}$. Following the principle of the minimum of total potential energy, $\delta\Pi = 0$ with respect to $\delta\boldsymbol{q}$ and using the conditions (1), one obtains Euler equation on L:

$$\frac{\partial L}{\partial \boldsymbol{q}} - \frac{\partial}{\partial z}\frac{\partial L}{\partial \dot{\boldsymbol{q}}} = 0 \tag{3}$$

In rational mechanics, L is named Lagrange's function, and (3) Lagrange's equation. Then we construct the Hamilton function H ($\boldsymbol{p}$, $\boldsymbol{q}$) through the Legendre's transformation:

$$\boldsymbol{p} = \frac{\partial L(\boldsymbol{q}, \dot{\boldsymbol{q}})}{\partial \dot{\boldsymbol{q}}}$$
$$H(\boldsymbol{p}, \boldsymbol{q}) = \boldsymbol{p}^T \dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}) \tag{4}$$

From (3) and (4), one deduces immediately the canonical equations of Hamilton:

$$\frac{\partial H}{\partial \boldsymbol{q}} = -\frac{\partial L}{\partial \boldsymbol{q}} = -\dot{\boldsymbol{p}} \quad \frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}} \tag{5}$$

$\boldsymbol{q}$ and $\boldsymbol{p}$ are dual conjugate variables. Differently from rational mechanics these two variables represent respectively the displacement vector and the normalised stress vector.

3 BUI'S FORMALISM; CAUCHY'S PROBLEM IN ELASTICITY

Bui [1] has solved the Cauchy problem in elasticity, by seeking the variations of the mechanical quantities ($\boldsymbol{q}$ as a displacement vector, $\boldsymbol{p}$ as a traction vector) at an arbitrary front in the solid when it virtuy moves from an initial position Γ_t to a neighbour position Γ_{t+dt}, where t defines the movement of the front in the solid. This approach leads to an explicit system of first-order differential equations.

Let consider a domaine divided into two parts Ω and Ω_t by a contour Γ_t. Suppose that mechanical fields are known at the interior of the contour; consequentley $\boldsymbol{q}$ and $\boldsymbol{p}$ are known at the contour Γ_t. Suppose $\boldsymbol{q}'$ a virtual compatible displacement. The virtual work principle leads to:

$$\int_{\Omega_t} \nabla \boldsymbol{q} . \boldsymbol{\Lambda} . \nabla \boldsymbol{q}' d\Omega = \int_{\Gamma_t} \boldsymbol{p} \cdot \boldsymbol{q}' d\Gamma \tag{6}$$

$\boldsymbol{\Lambda}$ is the elastic tensor. Let consider now an evolution of Γ_t to Ω_t, i.e. at $t+dt$, the contour Γ_t reaches Γ_{t+dt}. It's suitable to consider that Γ_{t+dt} se déduced from Γ_t by moving following the normal to Γ_t, with a quantity $\psi \boldsymbol{n} dt$, $\boldsymbol{n}$: is a unit vector normal to the contour and ψ: is a positif scalar field describing the velocity of the contour evolution. The variations of (6) with respect to dt gives:

$$\frac{d}{dt}\int_{\Omega_t}\nabla \boldsymbol{q}.\boldsymbol{\Lambda}.\nabla \boldsymbol{q}'\,d\Omega = \frac{d}{dt}\int_{\Gamma_t}\boldsymbol{p}\cdot\boldsymbol{q}'\,d\Gamma \tag{7}$$

If introducing the following notations of tangential operators :

$$grad_{\Gamma}(\cdot) := \nabla(\cdot) - \boldsymbol{n}\frac{\partial}{\partial n}(\cdot) \quad div_{\Gamma}(\cdot) := div(\cdot) - \boldsymbol{n}\cdot\frac{\partial}{\partial n}(\cdot) \tag{8}$$

equation (7) leads to:

$$\int_{\Gamma_t}\nabla \boldsymbol{q}.\boldsymbol{\Lambda}.\nabla \boldsymbol{q}'\,\psi d\Gamma - \int_{\Gamma_t}\left\{\frac{d\boldsymbol{p}}{dt} + div_{\Gamma}(\psi\boldsymbol{n})\boldsymbol{p}\right\}\cdot\boldsymbol{q}'\,d\Gamma - \int_{\Gamma_t}\psi\boldsymbol{p}\cdot\frac{\partial \boldsymbol{q}'}{\partial n}\,d\Gamma = 0 \tag{9}$$

After rearrangement and part integration one can deduce the following différential equations:

$$\begin{aligned}\frac{d\boldsymbol{q}}{dt} &= \boldsymbol{B}_q(\boldsymbol{q},\boldsymbol{p},\psi)\\ \frac{d\boldsymbol{p}}{dt} &= \boldsymbol{B}_p(\boldsymbol{q},\boldsymbol{p},\psi)\end{aligned} \tag{10}$$

$\boldsymbol{B}_q$ and $\boldsymbol{B}_p$ are expreced as function of quantities définied on the contour Γ_t. Their explicite expressions are given in the reference [1].

4 UNIFIED DESCRIPTION OF THE TWO FORMALISMS.

Here we describe a formalism unifying the two precedents within the frame of minimization of the total potential energy of the structure.

4.1 : HALMILTON PRINCIPAL WRITING AS VARIATION OF TOTAL POTENTIAL ÉNERGY. Following (2) et (4), the total potential energy is written:

$$\Pi = \int_{z_1}^{z_2} L\,dz = \int_{z_1}^{z_2}\left(\boldsymbol{p}\cdot\dot{\boldsymbol{q}} - H\right)dz \tag{11}$$

u is a parameter describing the solid's evolution. The description of a solid between an évent a and an évent b *could be done under* parametrical form of 7 functions in 2D media: twodisplacements $\boldsymbol{q}(u)$, three normalised stresses $\boldsymbol{p}(u)$ and one coordinate $z(u)$. Consider u_1 and u_2 as values of u corresponding to évents a and b. For $z_1=u_1$ et $z_2=u_2$, the total potential energy is re-written:

$$\Pi(u) = \int_{u_1}^{u_2}\left(\boldsymbol{p}\cdot\frac{\partial \boldsymbol{q}}{\partial u} - H\frac{\partial z}{\partial u}\right)du \tag{12}$$

And its variation becomes:

$$\delta\Pi = \frac{\partial\Pi}{\partial u}\delta u = \left\{\int_{u_1}^{u_2}\left[\frac{\partial \boldsymbol{p}}{\partial u}\cdot\frac{\partial \boldsymbol{q}}{\partial u} - \frac{\partial \boldsymbol{p}}{\partial u}\cdot\frac{\partial \boldsymbol{q}}{\partial u} - \frac{\partial H}{\partial u}\cdot\frac{\partial z}{\partial u} + \frac{\partial H}{\partial u}\frac{\partial z}{\partial u}\right]du + \left[\boldsymbol{p}\cdot\frac{\partial \boldsymbol{q}}{\partial u} - H\frac{\partial z}{\partial u}\right]_{u_1}^{u_2}\right\}\delta u \tag{13}$$

One notes :

$$\frac{\partial \boldsymbol{q}}{\partial u}\delta u = \delta\boldsymbol{q};\quad \frac{\partial \boldsymbol{p}}{\partial u}\delta u = \delta\boldsymbol{p};\quad \frac{\partial H}{\partial u}\delta u = \delta H;\quad \frac{\partial z}{\partial u}\delta u = \delta z \tag{14}$$

When u represents the coordonate z, (13) is written as follow:

$$\delta\Pi = \int_{z_1}^{z_2}\left[\dot{\boldsymbol{q}}.\delta\boldsymbol{p} - \dot{\boldsymbol{p}}.\delta\boldsymbol{q} - \delta H + \dot{H}\delta z\right]dz + \left[\boldsymbol{p}\cdot\delta\boldsymbol{q} - H\delta z\right]_{z_1}^{z_2} \tag{15}$$

So we have δΠ divided into two parts, the first one is an integral; the second one, is in the square bracket.

4.2: Application to the two-end problem. Consider now the variation of $\boldsymbol{q}$ and z are zero at *z1* and *z2*, $\delta\boldsymbol{q} = \boldsymbol{0}$ et $\delta z = 0$. This means we have fixed boundaries and fixed displacement boundary conditions at the two ends, so we have got the so-called two end point problem. In this case, the quantities in the square bracket equation (15) vanish. According to the principle of minimum total potential energy, we directly obtain the canonical equations of Hamilton. This is the problem resolved by the formalism of Zhong.

$$\delta\Pi = \int_{z_1}^{z_2} \left[\dot{\boldsymbol{q}}.\delta\boldsymbol{p} - \dot{\boldsymbol{p}}.\delta\boldsymbol{q} - \delta H + \dot{H}\delta z\right] dz = 0 \tag{16}$$

δH beeing: $\delta H = \dfrac{\partial H}{\partial \boldsymbol{q}}\delta\boldsymbol{q} + \dfrac{\partial H}{\partial p}\delta\boldsymbol{p} + \dfrac{\partial H}{\partial z}\delta z$, one obtains :

$$\delta\Pi = \int_{z_1}^{z_2} \left[\dot{\boldsymbol{q}}.\delta\boldsymbol{p} - \dot{\boldsymbol{p}}.\delta\boldsymbol{q} - \frac{\partial H}{\partial \boldsymbol{q}}\delta\boldsymbol{q} - \frac{\partial H}{\partial \boldsymbol{p}}\delta\boldsymbol{p} - \frac{\partial H}{\partial z}\delta z + \dot{H}\delta z\right] dz = 0 \tag{17}$$

This equation is available for arbitrary $\delta\boldsymbol{q}$, $\delta\boldsymbol{p}$ and δz. Consequently we deduce:

$$\frac{\partial H}{\partial z} = \dot{H}; \quad \frac{\partial H}{\partial \boldsymbol{p}} = \dot{\boldsymbol{q}}; \quad \frac{\partial H}{\partial \boldsymbol{q}} = -\dot{\boldsymbol{p}} \tag{18}$$

We find here Hamilton canonical equations.

4.3: APPLICATION TO CAUCHY'S PROBLEM. Now consider a natural evolution of the structure, this means that the Hamilton canonical equations are satisfied every where in the structure, but with possible variations of (***q***, *z*) at *z*=*z*1 and *z*=*z*2. In this case, we have no fixed boundaries nether fixed boundary conditions at the two ends but we have *natural* evolution everywhere, this is the so called Cauchy problem. In this case, the integral in equation (15) vanishes i.e.:

$$\delta\Pi = \boldsymbol{p}_2 \cdot \delta\boldsymbol{q}_2 - H_2 \cdot \delta z_2 - \boldsymbol{p}_1 \cdot \delta\boldsymbol{q}_1 + H_1 \cdot \delta z_1 \tag{19}$$

For a small displacement of events *a* and *b*, the variation of the total potential energy is:

$$\delta\Pi = \delta\boldsymbol{q}_1 \cdot \frac{\partial\Pi}{\partial \boldsymbol{q}_1} + \delta z_1 \frac{\partial\Pi}{\partial z_2} + \delta\boldsymbol{q}_2 \cdot \frac{\partial\Pi}{\partial \boldsymbol{q}_2} + \delta z_2 \frac{\partial\Pi}{\partial z_2} \tag{20}$$

The variables $\boldsymbol{q}_1$, z_1, $\boldsymbol{q}_2$, z_2 are indépendant. By identification between (19) and (20), we have got the Hamilton-Jacobi equations:

$$\frac{\partial\Pi}{\partial \boldsymbol{q}_2} = \boldsymbol{p}_2 \quad \frac{\partial\Pi}{\partial z_2} = -H_2 \quad \frac{\partial\Pi}{\partial \boldsymbol{q}_1} = -\boldsymbol{p}_1 \quad \frac{\partial\Pi}{\partial z_1} = H_1 \tag{21}$$

This is the problem resolved by Bui. We know that the Hamilton canonical equations and the Hamilton-Jacobi equations are equivalent. So we can say the formalism of Zhong and that of Bui are equivalent in the differential point of view, even they look quite different.

Now, dealing with Bui's formalism, it's obvious tahat the virtual work principle (6) could be written as a total potetial energy by replacing $\boldsymbol{q}'$ by virtual displacements $\delta\boldsymbol{q}$. Note that $d\Omega = d\Gamma dt$, one writes:

$$\delta\left[\int_t \frac{1}{2}\int_{\Gamma_t} \nabla\boldsymbol{q}.\boldsymbol{\Lambda}.\nabla\boldsymbol{q}\, d\Gamma dt - \int_{\Gamma_t} \boldsymbol{p}\cdot\boldsymbol{q}\, d\Gamma\right] = \delta\Pi = 0 \tag{22}$$

If we define:

$$L = \frac{1}{2}\int_{\Gamma_t} \nabla \boldsymbol{q}.\boldsymbol{\Lambda}.\nabla \boldsymbol{q} d\Gamma - \frac{d}{dt}\int_{\Gamma_t} \boldsymbol{p}\cdot\boldsymbol{q} d\Gamma$$

equation (22) becomes:

$$\delta\int_t L dt = 0 \qquad (23)$$

The partial derivation of (6) with respect to *t,* which represents the variation of virtual works due to virtual displcements during the evolution of the contour is equivalent to equation (23) if we consider a natural evolution.

5 HAMILTONIAN FORMALISM APPLIED TO FRACTURE MECHANICS

Return back now to fracture mechanics. We can actually write the equations governing the crack tip fields under form of (5). The main idea is to consider one coordinate in the polar system as "time" and take the total potential energy as the Lagrange function. For example, we can consider the radial coordinate r or the angular coordinate θ as time and take the variational principles established in continuum mechanics as the Hamilton variational principle. Then all the procedures currently used in rational mechanics can be translated into continuum mechanics.

By using this method, we have resolved some concrete problems. Some of them have been solved previously and some not yet. For example, we can calculate the stress singularities for an interfacial crack between two elastic and isotropic materials with the present method, the results are completely identical as those obtained by using the well-known theoretical formula. Similar example is a crack tip normally touching an interface has been resolved [3]. For a crack in a generally anisotropic material, we obtained identical near tip field as the theoretical results [4].The comparison shows no difference between these two stress distributions. Two other examples consist in finding stress singularities near a notch tip formed from two generally anisotropic materials and stress singularities near an inclined crack tip touching an interface between two generally anisotropic materials [4]. Asymptotic analysis for cracks in a bending plate formed by several anisotropic materials were also dealt with [5]. From this work, we see that the present method is particularly efficient for resolving multi-material problems. This is because the selected duals variables are continuous across all the interfaces. So the multi-material problem can be resolved as a single material problem through the construction of the transfer matrix.

6. CONCLUSION

In this paper, we give a new Hamiltonian formalism resolving simultaneously the two-end problem and the problem of Cauchy and as a consequence, showing the relationship between the formalisms of Bui and Zhong which look so different. The key idea is to write the total potential energy of a solid as an integral along a special axis z , then over a section S normal to it . Using integration by part, the variation of the total potential energy can be written as two parts [see (15)]. The first part is an integral along z, and the second one is an integrated quantity depending on the two ends z_1 and z_2. For the two end problem, the displacements are imposed at the two ends; so their variations vanish. According to the minimum principle of the total potential energy, the canonical equations of Hamilton are immediately obtained. [see (16)~(18)]. On the other hand, for a natural evolution of the structure (i.e., the canonical equations of Hamilton are satisfied everywhere in the solid), but with possible variations of the two ends, the first part in the variation of the total potential energy vanishes [see (19)]. This corresponds to the Cauchy problem in elasticity. In this case, the equations of Hamilton-Jacobi can be deduced [see (21)]. Since the

canonical equations of Hamilton and the equations of Hamilton-Jacobi are fundamentally equivalent, we can see that the formalisms of Bui and Zhong are equivalent too.
Zhong's formalism has been successfully applied to Fracture Mechanics in order to determine the asymptotic mechanical fields near the crack tip (Li and Recho, [3]). This work has shown that the Hamiltonian approach provides a systematic method in asymptotic analysis near the crack tip. It leads to a first order differential equation system, which is easy to deal with. We insist on the fact that this approach is not only a new formalism other than the traditional methods, but it can be used as a powerful tool in asymptotic analysis of the fracture mechanics. By using this new approach, we have successfully investigated some problems that may be difficult for traditional Airy function method. We believe that a large domain can be found in applying this new approach into fracture mechanics.

References

[1] Bui, H.D, Introduction aux problèmes inverses en mécaniques des matériaux, Editions Eyrolles, 1993
[2] Zhong W.X., A new systematical methodology in elasticity theory, Dalian Science & Technology University Press, 1995
[3] Li, J. , Recho, N., Méthodes asymptotiques en mécanique de la rupture, Editions Hermes Science, 2002
[4] Li, J., Zhang, X.B., Recho, N., Stress singularities near the Tipp of a two-dimensional notch forded from several anisotropic anisotropic materials, Int. J. Fract. 107, 379-395, 2001
[5] Li, J., Singularity analysis of near-tip fields for notches formed from several anisotropic plates under bending, Int. J. Solids & Struct., 39, 5767-5785, 2002