A MIXED APPROACH TO COHESIVE CRACK PROPAGATION

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ABSTRACT

A mixed variational formulation with discontinuous displacements and continuous tractions is proposed for the analysis of cohesive-crack propagation in elastic media. Such a formulation gives rise to a peculiar finite–element discretization scheme that is based on the Johnson-Mercier composite elements as to the stresses. Numerical results concerning classical benchmark problems are proposed and need for future developments highlighted, including the analysis of elastic-plastic and incompressible media.

1 INTRODUCTION

A new approach for cohesive crack propagation in elastic media is presented. Numerical methods for such kind of problems should be able to resolve discontinuous displacement fields as well as continuous tractions. Among pioneering contributions appeared up to the mid–nineties, the contribution [5] presented a re–meshing approach capable of following post–bifurcation and post–peak regimes, [6] introduced a stable, mesh–insensitive approach for the resolution of localization lines and [10] used a mixed formulation for the analysis of crack propagation in concrete specimens. More recently, several research groups all over the world have investigated the topic of cohesive–crack propagation in elastic media, mostly relying on the strong–discontinuity concept originally conceived by Simo and coworkers [14] and further extended in [13, 1, 3], among others. As to the numerical approximation of the continuous problem, the embedded–discontinuity approach and the extended–finite element method have emerged as the most appealing ones, see [8] for a detailed comparison of the potentials of the two approaches.
The distinctive feature behind our technique is the development of a novel and enriched version of the truly mixed Hellinger–Reissner variational principle [4] that induces inherently continuous tractions, say stresses $\underline{\underline{\sigma}}$ belong to the anisotropic space $H(\text{div}, \Omega)$, and discontinuous displacements, say $\underline{u} \in L^2(\Omega)$, Ω being the domain of the structure. In this respect, therefore, our approach seems to be ideally tailored for crack evolution problems as will be also elucidated in the sequel of the paper. Interface softening laws between adjacent elements are added in a natural way wherever the crack is evolving.

2 VARIATIONAL FORMULATION

Let $\Omega \in R^2$ and $\partial\Omega$ denote the domain of the system and its regular boundary, respectively. In the sequel, $\underline{\underline{\sigma}}$ and $\underline{\underline{\tau}}$ shall denote the unknown and test stress fields, respectively, $\underline{u}$ and $\underline{v}$ the unknown and test displacement fields, respectively, $\underline{\underline{\underline{\underline{C}}}}$ the fourth order elasticity tensor that enjoys the pointwise stability property and $\underline{g}$ the square integrable vector body load. Displacement and stress functional spaces to be used next respectively read

$$W = \left[L^2(\Omega)\right]^2, \tag{1}$$

$$H = H(\underline{\mathrm{div}}\,;\Omega) = \left\{\underline{\underline{\tau}} : \tau_{ij} = \tau_{ji},\ \tau_{ij} \in L^2(\Omega),\ \underline{\mathrm{div}}\ \underline{\underline{\tau}} \in W\right\}, \tag{2}$$

The modified Hellinger–Reissner variational formulation for elastic media in the presence of cracks eventually reads: find $(\underline{\underline{\sigma}}, \underline{u}) \in H \times W$ such that

$$\left\{\begin{aligned} &\int_\Omega \underline{\underline{\underline{\underline{C}}}}^{-1} \underline{\underline{\sigma}} : \underline{\underline{\tau}} dx + \int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\tau}} \cdot \underline{u} dx = \int_\Gamma [\![\underline{u}]\!] \cdot \left(\underline{\underline{\tau}} \cdot \underline{n}\right) dx, \quad \forall \underline{\underline{\tau}} \in H(\underline{\mathrm{div}}\,;\Omega), \\ &\int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\sigma}} \cdot \underline{v} dx = -\int_\Omega \underline{g} \cdot \underline{v} dx, \quad \forall \underline{v} \in W(\Omega). \end{aligned}\right. \tag{3}$$

A (rate–independent) cohesive–crack law relating the stress flux $\underline{\underline{\sigma}} \cdot \underline{n}$ and the opening displacement vector $[\![\underline{u}]\!]$ is then formally introduced along with its formal inverse as

$$\underline{\underline{\sigma}} \cdot \underline{n} = \mathcal{F}([\![\underline{u}]\!]),\ \ [\![\underline{u}]\!] = \mathcal{F}^{-1}(\underline{\underline{\sigma}} \cdot \underline{n}). \tag{4}$$

At this stage of the development, the properties of the operator $\mathcal{F}$ are left unspecified as is a discussion on the existence of its inverse and we proceed on a purely formal ground. By plugging Equation (4) into (3) one gets the following nonlinear variational problem: find $(\underline{\underline{\sigma}}, \underline{u}) \in H \times W$ such that

$$\left\{\begin{aligned} &\int_\Omega \underline{\underline{\underline{\underline{C}}}}^{-1} \underline{\underline{\sigma}} : \underline{\underline{\tau}} dx - \int_\Gamma \mathcal{F}^{-1}(\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \left(\underline{\underline{\tau}} \cdot \underline{n}\right) dx + \int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\tau}} \cdot \underline{u} dx = 0, \quad \forall \underline{\underline{\tau}} \in H(\underline{\mathrm{div}}\,;\Omega), \\ &\int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\sigma}} \cdot \underline{v} dx = -\int_\Omega \underline{g} \cdot \underline{v} dx, \quad \forall \underline{v} \in W(\Omega). \end{aligned}\right. \tag{5}$$

There emerge the following peculiarities of the proposed approach:

- the effect of the propagating crack turns out to be a modification of the complementary elastic energy that is replaced by its cracked–medium counterpart, i.e.

$$\underbrace{\int_\Omega \underline{\underline{\underline{\underline{C}}}}^{-1} \underline{\underline{\sigma}} : \underline{\underline{\tau}} dx - \int_\Gamma \mathcal{F}^{-1}(\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \left(\underline{\underline{\tau}} \cdot \underline{n}\right) dx}_{\text{elastic cracked medium}} \quad \leftarrow \quad \underbrace{\int_\Omega \underline{\underline{\underline{\underline{C}}}}^{-1} \underline{\underline{\sigma}} : \underline{\underline{\tau}} dx}_{\text{elastic uncraked medium}} \quad ,$$

- a remarkable symmetry arises as to the enforcement of the constitutive laws: both the bulk elastic law and the nonlinear cohesive crack law are imposed in strong form a-priori giving rise to two energy contributions that depend on the stress and on the stress–flux respectively,

- the unknown and test stress fields $\underline{\underline{\sigma}}$ and $\underline{\underline{\tau}}$ belong to $H(\underline{\mathrm{div}}\,;\Omega)$, ensuring a–priori the continuity of the stress–flux.

3 FINITE ELEMENT APPROXIMATION

To interpolate the stress field, the element of Johnson and Mercier [9] is introduced as one of the very few capable of passing the inf–sup condition in a truly mixed setting when coupled with element–wise linear, globally discontinuous displacements. Its usage for hardening-plasticity plane problems is suggested in [7] that also provides theoretical results on convergence rates, capability of passing the inf–sup condition and a-posteriori error estimates.

Each triangle K of the mesh is further subdivided into 3 sub–triangles T_i, so as to define the Johnson–Mercier stress space as:

$$JM(K) = \{\underline{\underline{\sigma}}|\ \underline{\underline{\sigma}} \in H(\underline{\mathrm{div}}\ ;K),\ \underline{\underline{\sigma}}|_{T_j} \in [P_1(T_j)]_s^{2\times 2}\ ,\ j = 1,2,3\}, \tag{6}$$

where $P_1(T_j)$ is the space of the polynomials of degree ≤ 1 on T_j. An element $\underline{\underline{\sigma}}$ of $JM(T)$ is uniquely determined by the following 15 degrees of freedom [9]:

$$\int_{e_i} (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \underline{w} ds, \quad \forall \underline{w} \in (P_1(e_i))^2, \quad i = 1,2,3, \tag{7}$$

$$\int_T \underline{\underline{\sigma}} : \underline{\underline{w}} dx, \qquad \forall \underline{\underline{w}} \in (P_0(T))_s^{2\times 2}, \tag{8}$$

where e_i denotes the i–th edge of the triangular element. Stresses are then approximated in the space

$$H_h = \left\{\underline{\underline{\sigma}}_h \in H(\mathrm{div}, \Omega),\ \underline{\underline{\sigma}}_h|_K \in JM(K)\right\}. \tag{9}$$

As to the displacements, one adopts an element–wise linear, globally discontinuous approximation, i.e. displacements are approximated in the space

$$W_h = \left\{\underline{v}_h \in W :\ \underline{v}_h|_K \in [P_1(K)]^2\right\}, \tag{10}$$

The discrete variational formulation therefore reads: find $(\underline{\underline{\sigma}}_h, \underline{u}_h) \in H_h \times W_h$ and $\Gamma_A \cup \Gamma_B \cup \Gamma_C \equiv \Gamma$ such that

$$\left\{\begin{aligned} &\int_\Omega \underline{\underline{\underline{\underline{C}}}}^{\,-1} \underline{\underline{\sigma}}_h : \underline{\underline{\tau}}_h dx - \int_{\Gamma_B} \mathcal{C}_{11}^{-1} \left(\underline{\underline{\sigma}}_h \cdot \underline{n}\right)_\perp \left(\underline{\underline{\tau}}_h \cdot \underline{n}\right)_\perp dx\ + \\ &\qquad + \int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\tau}}_h \cdot \underline{u}_h dx = \int_{\Gamma_B} \mathcal{C}_{11}^{-1} \left(\underline{\underline{\sigma}}_h \cdot \underline{n}\right)_\perp^* \left(\underline{\underline{\tau}}_h \cdot \underline{n}\right)_\perp dx, \quad \forall \underline{\underline{\tau}}_h \in H_h, \\ &\int_\Omega \underline{\mathrm{div}}\ \underline{\underline{\sigma}}_h \cdot \underline{v}_h dx = -\int_\Omega \underline{g} \cdot \underline{v}_h dx, \qquad\qquad \forall \underline{v}_h \in W_h(\Omega). \end{aligned}\right. \tag{11}$$

where $\mathcal{C}_{11}^{-1}$ is the pure mode I compliance.

4 NUMERICAL STUDIES

The methodology set forth in the preceding sections is applied to the numerical test of the specimen shown in Figure 1 where geometric dimensions in millimiters and one of the adopted meshes are given. The very same problem has been studied in [11] and originally in [5]. The physical properties are as follows: $E = 31370$ N/mm^2, $\nu = 0,2$, $\sigma_t^* = 4.4$ N/mm^2, $G_{IC} = 170$ J/m^2. G_{IC} represents the critical fracture energy in plane stress, i.e. the area under the cohesive curve. The relationship between G_{IC} and $\mathcal{C}_{11}$ reads

$$G_{IC} = \frac{(\sigma_t^*)^2}{2\ \mathcal{C}_{11}},$$

that implies for the case at hand $\mathcal{C}_{11} = 57$ N/mm^3. With such a choice, the maximum crack opening above which no cohesion is left amounts to $[\![\underline{u}]\!]_{\perp\max} = 0.15$ mm. As confirmed by experimental evidence, the geometry of the specimen under investigation is such to induce a fragile behavior with a fast propagation of the crack front that determines structural collapse well before the development of plasticity effects. Stress components corresponding to the peak–load are presented in Figures 2-3.

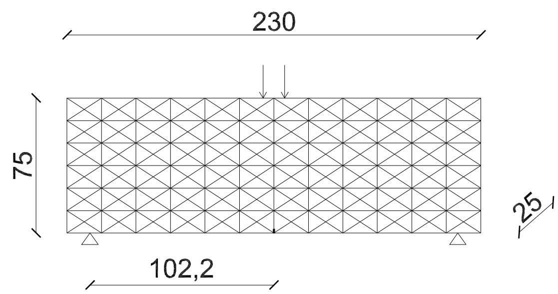

Figure 1: Specimen used in the numerical simulations

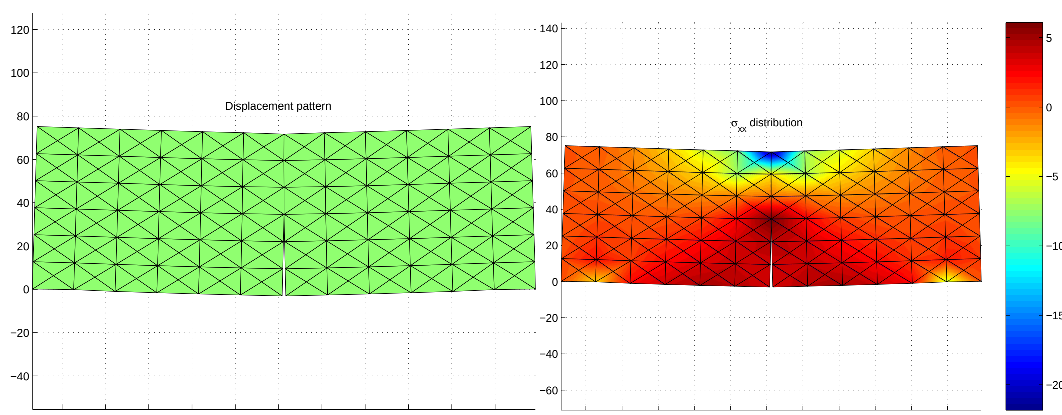

Figure 2: Deformed shape (left) - Stress component σ_{xx} (right)

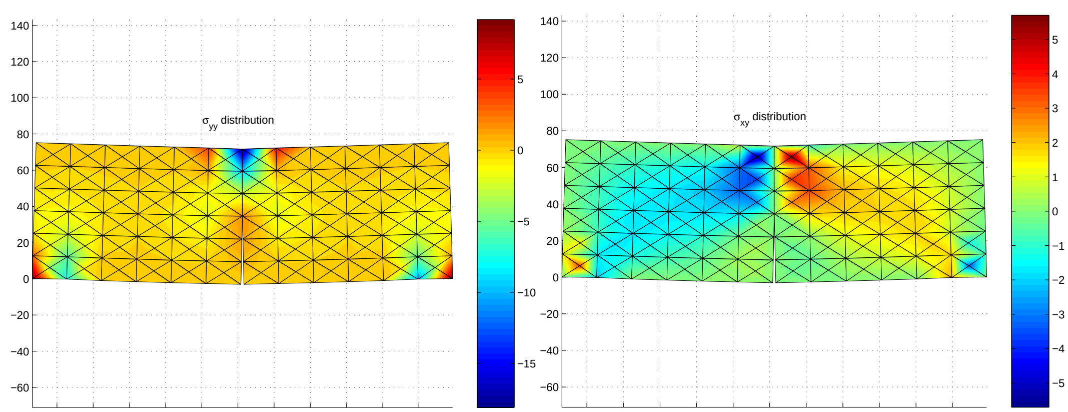

Figure 3: Stress component σ_{yy} (left) - Stress component σ_{xy} (right)

4.1 Convergence issues

A theoretical convergence analysis of the proposed approach is currently under study [**?**] along with considerations on the relevant inf–sup condition [4]. For this paper sake, convergence and mesh independence are assessed on a purely numerical basis by performing computations using three different meshes, i.e. (12×6, 24×12 and 36×18). Figure 4 presents the load–opening and load–vertical displacement diagrams for the three meshes used. It is apparent that results are nearly mesh insensitive and even a coarse mesh as the 12×6 one allows to capture the maximum load sustainable by the structure as well as both the ascending and descending branches of the load-opening diagram.

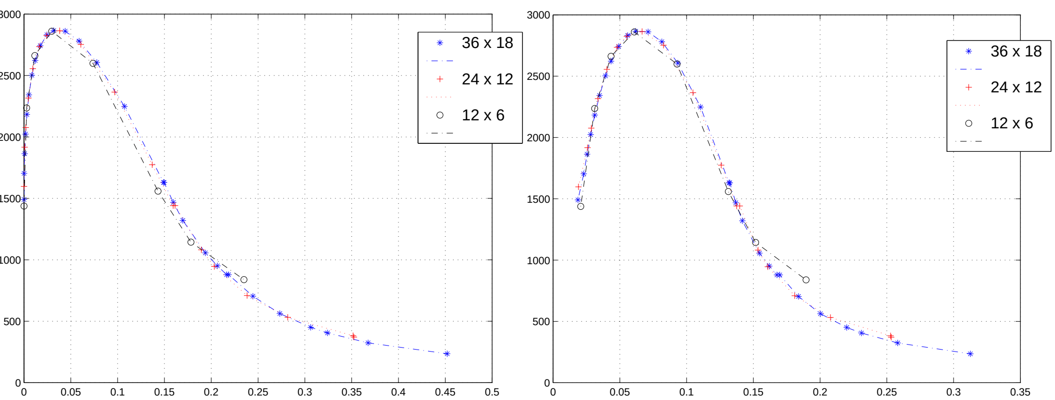

Figure 4: Load–opening curve (left) and load–vertical displacement curve (right for various meshes

5 CONCLUSIONS

A novel approach for the analysis of elastic media in the presence of cohesive cracks has been presented. The peculiarities of the proposed methodologies may be summarized as follows:

- the method rests on sound variational principles and is based on an extended Hellinger–Reissner formulation of truly mixed type;
- stresses are the primary (regular) variables and displacements the discontinuous Lagrange multipliers.

Ongoing research is focusing on the derivation of mesh manipulation procedures to allow the analysis of cohesive crack propagation along any path within the elastic medium. Furthermore methods for cohesive crack propagation in elastic–plastic and incompressible media are currently under development.

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