THE FRACTURE PROBLEM IN THE FRAMEWORK OF GENERALIZED STATISTICAL MECHANICS

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ABSTRACT

In order to study the relation between the ohmic resistance measured in a thin conducting ribbon and the length of a transversal cut, we employ a one-parameter deformed exponential and logarithm that were recently introduced in the framework of a generalized statistical mechanics. The analytical results have been compared with the data that was experimentally obtained and numerically computed with the boundary element method. Remarkably, the new deformed functions that interpolate between the standard functions and the power law functions, allow the best fit of the experimental data to be obtained for a wide range of the cut length.

1 INTRODUCTION

Non controlled fracture phenomena in bodies of any size and made of any type of materials is one of the most challenging puzzles in non equilibrium physics (Gerede [1]). In particular, concerning the problem of crack propagation in brittle material, for instance, glass or brittle plastic, some experimental (Fineberg et al. [2], Boudet et al. [3]) and theoretical papers (Abraham et al. [4], Omeltchenko et al. [5]) have been published but, until now, the accordance between experimental and theoretical results has been poorly satisfied (Sharon et al. [6], Ching [7]).

In experimental works, the fracture speed can be determined using an electronic indirect method. First, one needs to determine (experimentally and/or numerically) the electric resistance variations of a thin conductive film (eventually deposited on the specimen to be fractured) with respect to the fracture length. Than, one acquires a signal variable in time related to the rupture length: usually an out-of-balance voltage from a bridge circuit in which one arm contains the variable resistance. Finally, it is possible to calculate the fracture speed propagation by comparing the experimental data with an opportune model.

In this work we analyse the electric resistance variation undergone by a conductive ribbon whose initial width is progressively reduced by an artificial cut of a given length. The fracture length is hand controlled and measured. The corresponding electrical resistance values that are obtained up to complete rupture of the specimen, are compared with the numerical results obtained using the *boundary element method* (BEM). By employing the κ-exponential function (Kaniadakis [8,9]), we obtain the analytical relation between the electric resistance variations measured through the whole ribbon and the fracture length, by interpolating both the experimental and numerical data.

Remarkably, the κ-exponential is a smooth function interpolating between the classical exponential and the power law function. It can be successfully used in the study of problems showing a power law asymptotic behaviour.

2 EXPERIMENTAL SET UP

We now consider an ideal fracture experimental set up to determine the relation between the resistance and the cut length through the ribbon. The two-dimensional geometry of the sample is illustrated in the insert of figure 1. The sample is a ribbon with a rectangular shape of length $L_0 = FA$ and height $h = BB'$. The fracture is performed experimentally by cutting it with a sharp knife, step by step, along the median line FA. The solid line $FG = b$ represents the fractured line, while the dashed line $GA = s$ indicates the unbroken part of the ribbon, thus $L_0 = s + b$. The electrical resistance R of the conductive ribbon can be measured by applying a current generator in CD ad $C'D'$and measuring the voltage drop, that increases with the cut length.

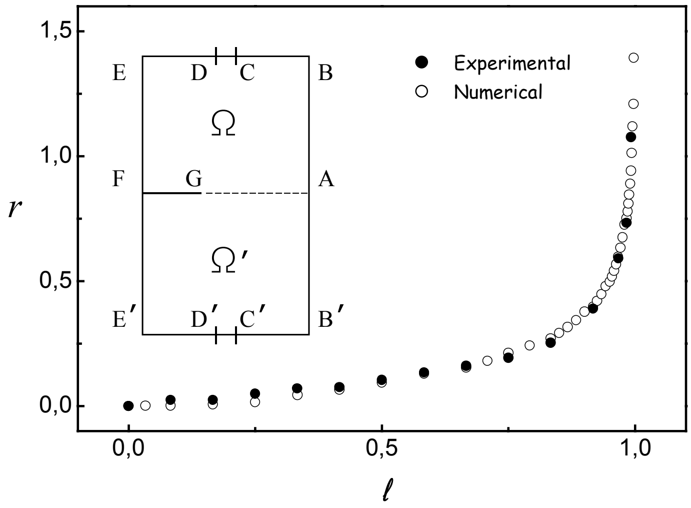

Figure 1. Plot of the rescaled resistance $r(\ell) = R(\ell) - R(0)$ data vs the normalized fracture length $\ell = b/L_0$ obtained experimentally (full circle) and numerically (open circle). The insert shows the geometrical setup of the ribbon.

The numerical simulation starts by considering the mirror symmetry with respect to the fracture line FGA. For simplicity, we consider only the rectangular domain $\Omega = ABCDEFGA$, with $\Gamma = \partial\Omega$ its boundary, and introduce the following notation: $\Gamma_1 = \Gamma_{CD}$ the contact junction of the ribbon with the external circuit, $\Gamma_2 = \Gamma_{GA}$ the length of the plate which is not yet broken, and $\Gamma_3 = \Gamma_{AB} \cup \Gamma_{BC} \cup \Gamma_{DE} \cup \Gamma_{EF} \cup \Gamma_{FG}$ the boundary of the sample without electric flux across it. The boundary of the region Ω thus results to be $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. Finally, let $U(x, y)$ be the electric potential normalized to unity; we assume the following *essential* condition: $U(x, y) = 1$ on Γ_1 and $U(x, y) = 0$ on Γ'_1 where $\Gamma'_1 = \Gamma_{C'D'}$. From symmetry considerations we pose $U(x, y) = 1/2$ on Γ_2, which is a *natural* condition.

The mathematical problem of the computation of the electric potential on the sample is described by the following Cauchy problem for the Laplace equation on the domain Ω:

$$\nabla^2 U(x,y) = 0 \ , \tag{1}$$

with $\partial U(x, y)/\partial \boldsymbol{n} = 0$ on Γ_3, where $\boldsymbol{n}$ is the outgoing unitary vector normal to the boundary Γ_3. In the stationary regime, the current flow through the ribbon is:

$$I = \int\limits_{\Gamma_1} \left| \frac{\partial}{\partial \boldsymbol{n}} U(x,y) \right| dl = \int\limits_{\Gamma_2} \left| \frac{\partial}{\partial \boldsymbol{n}} U(x,y) \right| dl \ . \tag{2}$$

Taking into account the boundary condition, we obtain the potential difference between the junctions Γ_1 and Γ_2:

$$V = U(x,y)|_{\Gamma_1} - U(x,y)|_{\Gamma_2} = 1/2 \ , \tag{3}$$

and, from Eqns (2) and (3) one has the expression of the resistance of the ribbon:

$$R = \left(2 \int\limits_{\Gamma_1} \left| \frac{\partial}{\partial \boldsymbol{n}} U(x,y) \right| dl \right)^{-1} \ . \tag{4}$$

Figure 1 shows the data of the rescaled resistance $r = R(\ell) - R(0)$ with respect to the normalized length of the fracture $\ell = b/L_0$, obtained both experimentally (full circle) and numerically (open circle).

2 κ-DEFORMED EXPONENTIAL AND LOGARITHM

Recently (Kaniadakis [8,9,10]), starting from an one parameter deformation of the statistical mechanics, which reduces to the ordinary Boltzmann-Gibbs theory as the deformation parameter κ approaches zero, it has been introduced a κ-deformed version of the exponential and the logarithmic functions, namely the κ-exponential and the κ-logarithm.

The κ-exponential is a continuous, one parameter function:

$$\exp_{\{\kappa\}}(x) = \left(\sqrt{1+\kappa^2 x^2} + \kappa\, x\right)^{1/\kappa} \ , \tag{5}$$

which conserves many properties of the standard exponential which we recall briefly.

The κ-exponential $\exp_{\{\kappa\}}(x) = \exp_{\{-\kappa\}}(x)$, is a positive definite and increasing function for $x \in \mathbb{R}$ and $\kappa \in (-1, 1)$, that reduces to the standard exponential for $\kappa \to 0$, $\exp_{\{0\}}(x) = \exp x$. Remarkably the κ-exponential decreases for $x \to -\infty$ and increases for $x \to +\infty$ with the same steepness: $\exp_{\{\kappa\}}(x) \exp_{\{\kappa\}}(-x) = 1$, and the scale law hold in the form $[\exp_{\{\kappa\}}(x)]^\lambda = \exp_{\{\kappa/\lambda\}}(\lambda x)$. Moreover, the asymptotic behaviours of the κ-exponential for small and large x are given, respectively:

$$\exp_{\{\kappa\}}(x) \underset{x\to 0}{\sim} \exp(x) \ , \qquad \exp_{\{\kappa\}}(x) \underset{x\to\pm\infty}{\sim} |2\kappa x|^{\pm 1/|\kappa|} \ . \tag{6}$$

The plot of the κ-exponential with $\kappa = 0.3$, compared with the classical exponential and the power function $ax^{1/\kappa}$ with $a = 0.182$, is shown in figure 2, in logarithmic scale. The same figure emphasizes the two asymptotic regions that show that the κ-exponential is a smooth function which interpolates between the classical exponential and the power function.

The inverse function, the κ-logarithm, can also be introduced:

$$\ln_{\{\kappa\}}(x) = \frac{x^\kappa - x^{-\kappa}}{2\,\kappa} \ . \tag{7}$$

This is a real and increasing function for $x > 0$, reducing, in the $\kappa \to 0$ limit, to the standard logarithmic: $\ln_{\{0\}}(x) = \ln x$. It satisfies the relation $\ln_{\{\kappa\}}(x) = \ln_{\{-\kappa\}}(x)$ and the following scaling law: $\ln_{\{\kappa\}}(x^\lambda) = \lambda \ln_{\{\lambda\kappa\}}(x)$. The asymptotic behaviours for small and large x are, respectively

$$\ln_{\{\kappa\}}(x) \underset{x\to 0^+}{\sim} -\frac{1}{|2\kappa|}\, x^{-|\kappa|} \ , \qquad \ln_{\{\kappa\}}(x) \underset{x\to+\infty}{\sim} \frac{1}{|2\kappa|}\, x^{|\kappa|} \ .$$

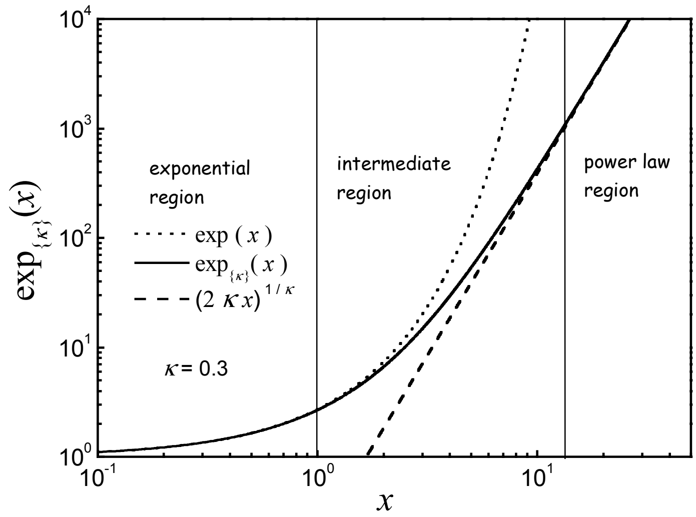

Figure 2. Comparison of the κ-exponential function (κ = 0.3) with the Classical exponential and the power function $ax^{1/\kappa}$ (a = 0.182).

The κ-entropy, defined by:

$$S_\kappa[f] = -k_{\mathrm{B}} \int f \, \ln_{\{\kappa\}}(f) \, d^3 v \quad , \tag{10}$$

reduces to the standard Boltzmann-Gibbs entropy $S_0[f] = -k_{\mathrm{B}}\int f \ln f \, d^3 v$ as $\kappa \to 0$. In (Kaniadakis [8]) it is shows that S_κ is obtained by a continuous deformation of S_0 and preserves its fundamental properties of concavity, additivity and extensivity. Starting from this entropy it is possible to construct a generalized statistical mechanics and thermodynamics, that have the same mathematical and epistemological structure of the standard Boltzmann- Gibbs theory.

According to the MaxEnt principle, the distribution function f obtained by optimising Eqn (10) with the constraints $\int f \, d^3 v = 1$ for the normalization and $\int E f \, d^3 v = < E >$ for the mean energy is given by

$$f = \alpha \, \exp_{\{\kappa\}} \left[-\frac{\beta}{\lambda} \, (E - \mu) \right] \quad , \tag{11}$$

and depends on the unspecified parameter β, that contains all the information about the temperature of the system. The parameters α and λ are related to κ by the relations.

$$\alpha = [(1-\kappa)/(1+\kappa)]^{1/2\kappa} \, , \qquad \text{and} \qquad \lambda = \sqrt{1-\kappa^2}. \tag{12}$$

We remark that the distribution (11) show an asymptotic long tail with a power law behaviour.

As shown in (Kaniadakis [8]), the origin of the deformation mechanism introduced by κ, emerges naturally within Einstein's special relativity. The value of the free parameter κ in particular depends on the light speed c. Only in the classic limit c $\to \infty$ the parameter κ approaches zero. Thus the κ-deformation is originated from the finite value of light speed and results to be a purely relativistic effect.

The cosmic rays represents the most important example of relativistic statistical system which manifestly violates the Boltzmann statistics. In (Kaniadakis [8]) it has been shown that the κ-statistics can predict very well the experimental cosmic ray spectrum. This is an important test for the theory because the cosmic rays spectrum has a very large extension (13 decades in energy and 33 decades in flux).

Let us consider now statistical systems (physical, natural, economical, etc.) in which is involved a limiting quantity like the light speed in the relativistic particle system. For these systems where the information propagates with finite velocity, it is reasonable suppose that the κ-deformation can appear, so that the κ-statistics results to be the most appropriate theory to describe these systems.

4 DATA ANALYSIS

Let us now introduce the rescaled length $l(r)$ = 1- $\ell(r)$ with $l(0)$ = 1 and $l(\infty)$ = 0, and take the following ansatz:

$$l(r) = \exp_{\{\kappa\}}\left(-\frac{r}{r_0}\right) \ , \tag{13}$$

where the t*emperature* r_0 and the deformed parameter κ are related in an unspecified way to the physical and geometrical properties of the ribbon and constitute a set of free parameters.

The determination of these parameters is well accomplished by matching Eqn (13) with the experimental data. If we look at the asymptotic expression of Eqn (13) for a large value of r, from Eqn (6) it follows that:

$$\ln l \simeq \frac{1}{\kappa} \ln r_\infty - \frac{1}{\kappa} \ln r \ , \tag{14}$$

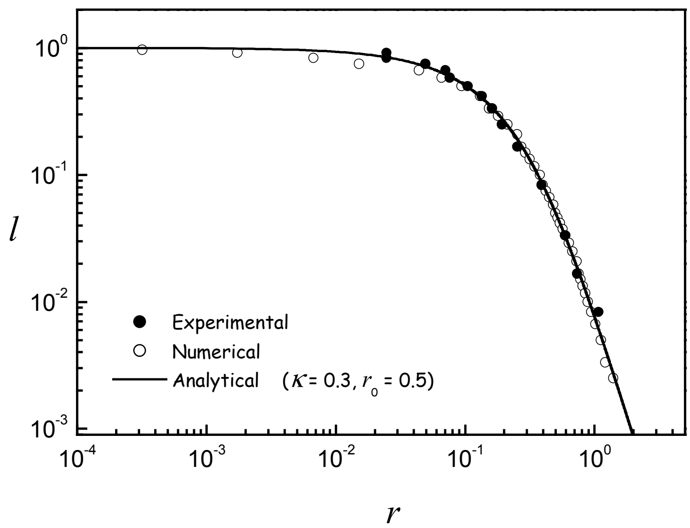

Figure 3. Best-fit of the fracture data using the κ-exponential distribution with the deformed parameter κ = 0.3 and *temperature* r_0 = 0.5.

where $r_\infty = r_0/2\kappa$ is a constant. Eqn (14) is a straight line whose slope is given by the inverse of the deformation parameter κ. On the other hand, the quantity r_∞, related to the *temperature* r_0, produces a translation of the curve along the rescaled length axis.

Figure 3 is a plot of Eqn (13) ($\kappa = 0.3$, $r_0 = 0.5$) in logarithmic scale, compared with the experimental and numerical results presented in figure 1. It shows an excellent agreement between the analytical and both numerical and experimental data for a wide range of values of the rescaled length $l(r)$. It should be noted that, for scale reasons, the first point of figure 1 corresponding to $l = 0$ has been removed.

We can remark that the relation obtained between the rescaled resistance and the cut length is not a consequence of the particular geometrical set up that was here adopted. In fact, it can be observed that whether the position of the cut line *FA*, than the position of the elements *CD* and *C'D'* where the current generator is applied are changed, or the length and height of the ribbon are changed the results are always qualitatively the same.

In conclusion the following remarks can be made. The experimental data confirm, as do the numerical results of BEM, that, in the situation in which a current flux goes through a thin conductor whose section in a point is shrunk by a running fracture, the relation between the resistance and the reduced section must be considered with care. The BEM algorithm intrinsically takes into account, in the resistance evaluation, the complete statistical distribution of the infinite paths the current can follow in the fracturing ribbon. From a statistical point of view, this means that the distribution of the measured $l(r)$ is the typical distribution of nonextensive statistical systems. We have shown that the relation between the electric resistance and the cut lengthdepends on the size of the transverse section of the conducting channel. From Eqn (13) we obtain for $s\to 0$ the following asymptotic behaviour $R(s) - R(0) \sim s^{-|\kappa|}$. This relationship between the resistance and the transverse section must be taken into account in the experimental works in which the speed of the fracture is determined through resistance measurements.

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