Residual Shear Stresses and K_{II} Computation

Weili Cheng[1] and Iain Finnie[2]

[1] BEAR, Inc. 2216 5th Street Berkeley, CA 74710-2217, USA

[2] Department of Mechanical Eng. University of California, Berkeley, CA 94720 USA

ABSTRACT

Legendre polynomials of second and higher orders have been widely used to describe residual normal stresses because of their unique properties that satisfy the force and moment equilibrium conditions. Unlike residual normal stresses, an expression for the shear stresses needs to satisfy the stress-free surface condition and the force equilibrium condition only. In this paper a general expression is derived for residual shear stresses. This allows an arbitrary shear residual stress to be presented explicitly in terms of polynomials, which is useful for both computation of K_{II} due to residual shear stresses and measurement of shear stress through the thickness.

1 INTRODUCTION

To calculate stress intensity factors due to residual stresses, it is necessary to ensure that the stress used in computation satisfies equilibrium conditions and boundary conditions. In this case Legendre polynomials of orders 2 and higher are often used for residual normal stresses. For mode II loading it is desirable to have a similar polynomial series for residual shear stresses. In this paper, we demonstrate that a complete set of orthogonal functions from the family of Jacobi polynomials satisfies the equilibrium condition and the boundary conditions for residual shear stresses. In the second part of the paper, a newly developed method based on FEM for K_I is extended to compute K_{II} due to residual shear stresses. After a brief description of the approach, the analysis for mode II loading is presented. Results for K_{II} due to residual shear stresses are then obtained.

2 A GENERAL EXPRESSION FOR RESIDUAL SHEAR STRESSES

When computing stress intensity factors, the stresses are usually expressed in terms of a polynomial. For residual normal stresses, Legendre polynomials of second or higher orders have been used to approximate an arbitrary residual stress field through the thickness because each term of the polynomials satisfies the force and moment equilibrium condition. However, to authors' knowledge such an expression has not yet been presented for residual shear stress. In the first part of the presentation a general expression for shear stresses through the thickness will be derived.

Consider a plate of thickness t with shear stress τ_{xy} along the x-direction, as shown in Fig. 1. To simplify the derivation that follows, we take the origin at the centerline of the plate and normalize the thickness by t/2. Since the shear stress always acts tangential to the surface, it must vanish at free surface, x = -1 and x = 1. The stress free boundary conditions at x = -1 and 1 suggest that $\tau_{xy}(x)$ may be expressed as

$$\tau_{xy}(x) = (1-x)(1+x)J(x) \tag{1}$$

where $J(x)$ is a function to be determined. Since the share stress must satisfy the force equilibrium condition

$$\int_{-1}^{1} \tau_{xy}(x)\,dx = \int_{-1}^{1} (1-x)(1+x)\,J(x)dx = 0, \tag{2}$$

$J(x)$ clearly can be constructed by a set of orthogonal functions with a weight function equal to (1-x)(1+x). In fact, $J(x)$ can be shown to belong to the family of Jacobi polynomials [1], which also include the Legendre polynomials. That is,

$$J_n(x) = \frac{(-1)^n}{2^n n!}(1-x^2)^{-1}\frac{\partial^n}{\partial x}\left[(1-x^2)^{1+n}\right] \tag{3}$$

In practice, the computation of the nth order polynomial J_n is more conveniently carried out by using the recurrence relation

$$J_n(2+n)n = (2n+1)(n+1)\,x\,J_{n-1} - n(n+1)\,J_{n-2}$$
$$with \tag{4}$$
$$J_0 = 1, \quad J_1(x) = 2x$$

Figure 2 shows the variations of $\tau_{xy}(x)$ for the first six terms of the polynomial series. It is seen that for J_0 = 1, the shear stress corresponds to a parabolic distribution produce by shear loads acting at ends of the plate. For J_i with i > 0, the resultant force over the thickness is always zero. It is noticed that J_i consist of a complete set of polynomials. Thus, an arbitrary shear stress may be expressed as

$$\tau_{xy}(x) = (1-x)(1+x)\sum_{i=0}^{n} \beta_i J_i(x) \tag{5}$$

where β_i are amplitude coefficients. From eqn (5) a residual shear stress can be obtained when the first term (i = 0) is omitted.

3 COMPUTATION OF STRESS INTENSITY FACTOR DUE TO SHEAR STRESSES

A newly developed method [2] based on FEM for K_I is extended to compute K_{II} due to a shear stress. Consider the configuration shown in Fig. 3. The displacement, $u(a,S)$, due to a crack of size a in the x-direction at a location S can be obtained by introducing a virtual force Q at S in the direction of $u(a,S)$. Following the approach used for K_I, we find an expression for mode II loading as

$$K_{II}(a) = \left[\frac{\partial u(a,S)}{\partial a}\right] \Big/ \left[\frac{\partial u^q(a,S)}{\partial a}\right] \tag{6}$$

in which $u^q(a,S)$ is the displacement in x-direction produced by the virtual force Q. The differentiation in eqn (6) may be computed using the Simpson’s rule, i.e.

$$\frac{\partial(u(a_i)}{\partial a} \approx \frac{u(a_{i+1}) - u(a_{i-1})}{a_{i+1} - a_{i-1}} \tag{7}$$

Results of K_{II} are then obtained for shear stresses given in the form of eqn (5) using a finite element mesh shown in Fig. 4.

4 REFERENCES

1. D. E. Johnson and J. R. Johnson, Mathematical Methods in Engineering and Physics, Prentice-Hall, Inc., Englewood Cliffs, NJ 07632, 1982.
2. W. Cheng and I. Finnie, Computation of Stress Intensity Factors for a 2-D body from Displacements at an Arbitrary Location, Int. J. of Fracture, 81, 259-267, 1996.

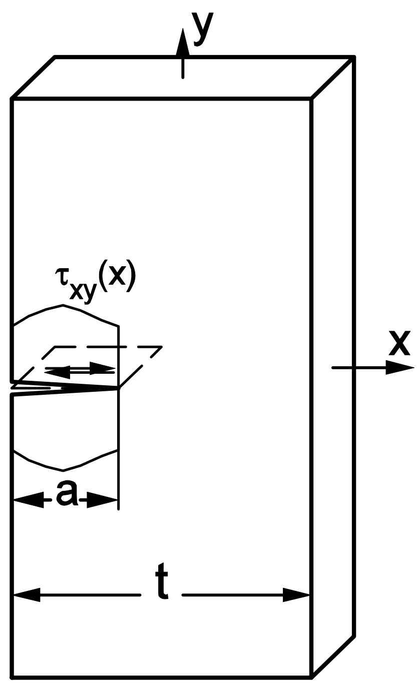

Figure 1: An edge-cracked plate subjected shear stress on the faces of the crack.

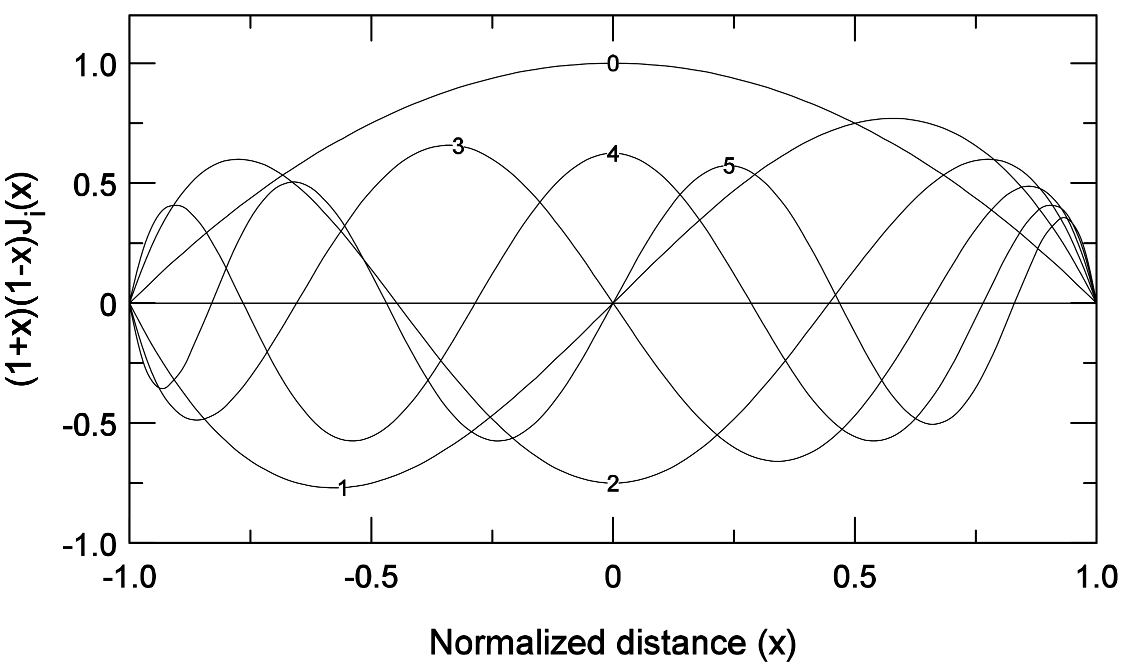

Figure 2: Plots of the first six terms of (1+x)(1-x) $J_i(x)$

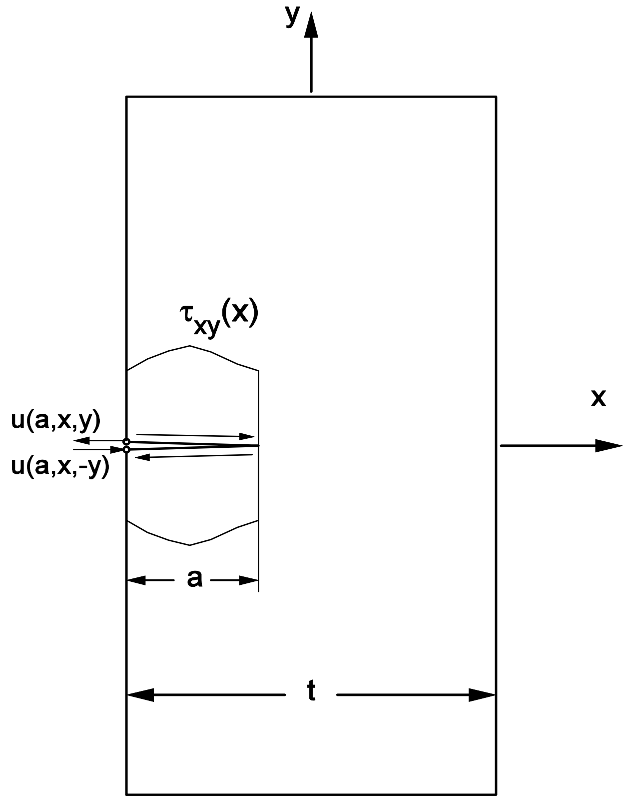

Figure 3: Computation of K_{II} using the displacements due to the shear stress on the crack faces and the virtual forces at the locations of the displacements.

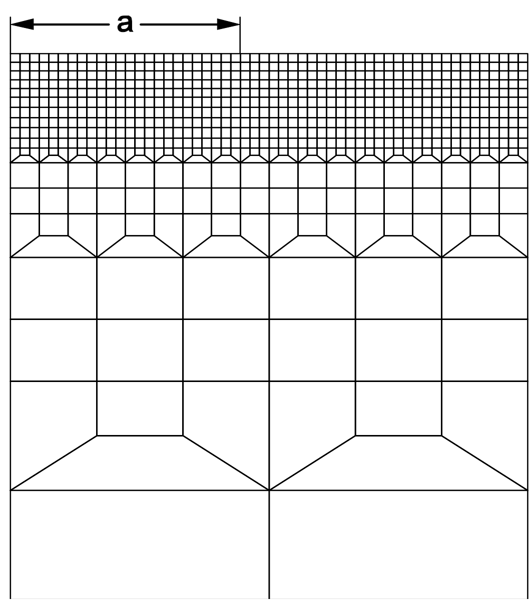

Figure 4: Finite element mesh for an edge-cracked plate with evenly spaced increments.