COSSERAT CONTINUUM MODEL OF WAVE PROPAGATION IN LAYERED MATERIALS

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ABSTRACT

We derive a model for the long wavelength behaviour of a layered material with smooth interfaces within the framework of Cosserat continuum. In such a continuum three types of waves exist: longitudinal, shear and rotational. The wave velocity in this approximation depends on the direction of propagation, but not on the absolute value of the wave vector. Only a longitudinal wave exists in the directions strictly perpendicular and parallel to the layering. This has a profound effect on the fracture propagation caused by an impact action perpendicular to the layering. The impact will be transferred from one layer to the other at the speed of the longitudinal wave. Thus, when the first layer breaks the longitudinal wave will transmit the load to the next one and so forth, without the energy dissipation that would be associated with the propagation of rotational and shear waves.

1. INTRODUCTION

In layered materials nonstandard terms of the continuum theory can be used to model the influence of the bending stiffness of each layer on the material's response. In this way a second (internal) length scale is introduced in addition to the global one defined, for example, by structural dimensions. In this case the internal lengths are simply the layer thicknesses. Note that, strictly speaking, a layered material can be modelled as a conventional orthotropic continuum only if the layer thickness is zero (or negligibly small as compared to a characteristic structural length). The standard continuum approach is inapplicable if the layer parallel shear modulus vanishes, i.e. if the layer interfaces are smooth.

Stress and moment stress localization occurring in the direction perpendicular to layering have been studied in connection with crack propagation in [1]. Wave propagation in the case of the Cosserat continuum for blocky structures has been investigated in [2]. The wave propagation in the case of a higher-order gradient continuum for granular materials has been analysed in [3,4]. In this paper this approach will be applied to layered materials and will be used to study the impact failure mechanism and offer a partial explanation for the ability of a karate expert to break a pile of slabs.

2. WAVE PROPAGATION IN THE LAYERED MATERIAL

The Lamé equations with dynamic terms for Cosserat continuum model of layered material with frictionless interfaces have the form (see [1, 4] for details)

$$A_{11}\frac{\partial^2 u}{\partial x^2} + A_{12}\frac{\partial^2 v}{\partial x \partial y} + f_x = \rho \ddot{u}, \tag{1}$$

$$A_{12}\frac{\partial^2 u}{\partial x \partial y} + G''\frac{\partial^2 v}{\partial x^2} + A_{22}\frac{\partial^2 v}{\partial y^2} - G''\frac{\partial \Omega_z}{\partial x} + f_y = \rho \ddot{v}, \tag{2}$$

$$B\frac{\partial^2 \Omega_z}{\partial x^2} + G''\frac{\partial v}{\partial x} - G''\Omega_z + m_z = \tilde{J}\ddot{\Omega}_z, \tag{3}$$

$$A_{11} = A_{22} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)}, \quad A_{12} = \frac{\nu E}{(1+\nu)(1-2\nu)}, \quad B = \frac{Eb^2}{12(1-\nu^2)}, \quad \tilde{J} = \rho\frac{b^2}{12}. \tag{4}$$

We have assumed here that x-axis of the Cartesian co-ordinate set (x,y) be directed parallel to the layers, the y direction is orthogonal to the layering, (u,v) is the displacement vector, Ω_z is an independent Cosserat rotation representing the rotation of the cross-section of the layer; (f_x, f_y) and m_z are body force and body moment respectively; E, ν are the layer Young's modulus, and Poisson's ratio, $G''=G=E/2(1+\nu)$, b is the layer thickness, ρ is the density and $\tilde{J}$ is the rotational inertia per unit volume.

In such a continuum three types of waves exist: longitudinal, shear and rotational, the latter being associated with the Cosserat rotation. For a particular case of zero body forces and body moment, ie $f_x=f_y=m_z=0$ we consider the propagation of harmonic waves of the form:

$$\begin{Bmatrix} u \\ v \\ \Omega_z \end{Bmatrix} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} e^{i\xi(x-v_{p_x}t)} \cdot e^{i\eta(y-v_{p_y}t)} = \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} e^{i(\xi x+\eta y-\omega t)}, \tag{5}$$

where ξ and η are wave numbers in the x and y (parallel and perpendicular to the layering) directions respectively, $\boldsymbol{v}=\{v_{px}, v_{py}\}$ the phase velocity, $\omega= \xi v_{px} +\eta v_{py}$ frequency.

Substitution (5) into (1)-(3) yields the following homogeneous system of equations:

$$A_{11}\xi^2 B_1 + A_{12}\xi\eta B_2 - \rho\omega^2 B_1 = 0, \tag{6}$$

$$A_{12}\xi\eta B_1 + G''\xi^2 B_2 + A_{22}\eta^2 B_2 + G'' i\xi B_3 - \rho\omega^2 B_2 = 0, \tag{7}$$

$$B\xi^2 B_3 - G'' i\xi B_2 + G'' B_3 - \tilde{J}\omega^2 B_3 = 0. \tag{8}$$

The system has non-trivial solution if its determinant vanishes:

$$\begin{vmatrix} A_{11}\xi^2 - \rho\omega^2 & A_{12}\xi\eta & 0 \\ A_{12}\xi\eta & G''\xi^2 + A_{22}\eta^2 - \rho\omega^2 & G'' i\xi \\ 0 & -G'' i\xi & B\xi^2 + G'' - \tilde{J}\omega^2 \end{vmatrix} = 0 \tag{9}$$

or, after the normalization,

$$\frac{E}{1+\nu} = 1, \quad A_{11} = A_{22} = \frac{1-\nu}{1-2\nu}, \quad A_{12} = \frac{\nu}{1-2\nu}, \quad B = \frac{b^2}{12(1-\nu)}, \quad G'' = 1/2 \tag{10}$$

$$-\rho^2\tilde{J}\omega^6 + \frac{\rho}{12}\frac{\left(-42\tilde{J}\nu + \rho b^2 + 18\tilde{J} + 24\tilde{J}\nu^2 - 2\rho b^2\nu\right)\xi^2 + 6(\nu-1)\left(2\rho\nu + 2\eta^2\tilde{J}\nu - \rho - 2\eta^2\tilde{J}\right)}{(1-2\nu)(1-\nu)}\omega^4 +$$

$$\frac{1}{24}\frac{\left(4\rho b^2\nu - 12\tilde{J}\nu^2 + 24\tilde{J}\nu - 12\tilde{J} - 3\rho b^2\right)\xi^4 + 2(\nu-1)\left(-6\rho\nu + 12\eta^2\tilde{J} + 6\rho + \rho b^2\eta^2\right)\xi^2 - 12\rho\eta^2(\nu-1)^2}{(1-2\nu)(1-\nu)}\omega^2 +$$

$$\frac{1}{24}\frac{-b^2(\nu-1)\xi^6 + 2b^2\xi^4\eta^2 - 12(\nu-1)\xi^2\eta^2}{(1-2\nu)(1-\nu)} = 0. \tag{11}$$

Bearing in mind that the validity of the Cosserat continuum theory is restricted to a long wave asymptotics, so that the layer thickness b to wave length L ratio must be $b/L<<1$, we now consider the limiting case for small $b\eta$ and $b\xi$. Assuming $b\eta<<1$, $b\xi<<1$, $\tilde{J}\sim\rho b^2$ the equation (11) can be simplified:

$$\rho^2\omega^4 - \frac{1-\nu}{1-2\nu}\rho\left(\xi^2+\eta^2\right)\omega^2 + \frac{1}{12(1-2\nu)}\xi^2\left(b^2\xi^4 + 12\eta^2\right) = 0. \tag{12}$$

The discriminant of the resulting bi-quadratic equation is

$$D = \frac{3(1-\nu)^2\left(\xi^2+\eta^2\right)^2 - \xi^2\left(b^2\xi^4+12\eta^2\right)(1-2\nu)}{3(1-2\nu)^2}. \tag{13}$$

In (13) the term $\xi^6 b^2 = \xi^2 b^2 \xi^4$ can be neglected with respect to the term $\xi^2\eta^2$ if $\eta \neq 0$ and with respect to the term ξ^4 if $\eta=0$. Then the discriminant (13) in the long wave asymptotics assumes the form:

$$D = \frac{(1-\nu)^2}{(1-2\nu)^2}\left(\xi^2+\eta^2\right)^2 - 4\frac{\xi^2\eta^2}{1-2\nu} > 0\cdot \tag{14}$$

With these assumptions, the dispersion relations are obtained as

$$\rho\omega_{1,2}^2 = \frac{1-\nu}{2(1-2\nu)}\left(\xi^2+\eta^2\right) \pm \frac{1}{2}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}\left(\xi^2+\eta^2\right)^2 - \frac{4\xi^2\eta^2}{1-2\nu}}. \tag{15}$$

Formula (15) gives two solutions for the square of the phase frequency ω. It is clear that because of the absence of the thickness of the layer b in (15), the frequency, given by a homogeneous function of the first order $\omega(k\xi,k\eta)=k\omega(\xi,\eta)$, is formally valid for all wave numbers even if $b\xi$ and $b\eta$ are not small. It also contains no information on the layer orientation in this approximation since wave numbers ξ and η can be interchanged without effect on the frequency.

For the further investigation of the wave propagation problem it is convenient to use the polar coordinates: $\xi=r\cos\varphi$, $\eta=r\sin\varphi$. Then (15) becomes

$$\sqrt{\rho}\,\omega_{1,2} = r\sqrt{\frac{1-\nu}{2(1-2\nu)} \pm \frac{1}{2}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2} - \frac{\sin^2 2\varphi}{1-2\nu}}}. \tag{16}$$

It appears from (16) that the frequency is proportional to the absolute value of the wave vector r

$$\sqrt{\rho}\,\omega_{1,2} \sim r\cdot \tag{17}$$

Next we find the components of the phase velocity $\boldsymbol{v}=\{v_{px}, v_{py}\}$. Since $\omega= \xi v_{px} + \eta v_{py}$ and

$$\omega(\xi,\eta) = \xi\frac{\partial\omega}{\partial\xi} + \eta\frac{\partial\omega}{\partial\eta} \tag{18}$$

one obtains that

$$v_{px} = \frac{\partial\omega}{\partial\xi},\ v_{py} = \frac{\partial\omega}{\partial\eta}\cdot \tag{19}$$

By differentiating (15) with respect to ξ and η, the components of the phase velocity can be found

$$\left(v_{px}\right)_{1,2} = \frac{1}{2\rho\omega_{1,2}}\xi\frac{1-\nu}{1-2\nu}\left(\pm 2\rho\omega_{1,2}^2 - \frac{2\eta^2}{1-\nu}\right)\left[\frac{(1-\nu)^2}{(1-2\nu)^2}\left(\xi^2+\eta^2\right)^2 - \frac{4\xi^2\eta^2}{1-2\nu}\right]^{-1/2}, \tag{20}$$

$$\left(v_{py}\right)_{1,2} = \frac{1}{2\rho\omega_{1,2}}\eta\frac{1-\nu}{1-2\nu}\left(\pm 2\rho\omega_{1,2}^2 - \frac{2\xi^2}{1-\nu}\right)\left[\frac{(1-\nu)^2}{(1-2\nu)^2}\left(\xi^2+\eta^2\right)^2 - \frac{4\xi^2\eta^2}{1-2\nu}\right]^{-1/2}. \tag{21}$$

In (20), (21) only positive values of the square roots are used, since the negative values would correspond to the waves travelling in the opposite direction.

The case of special interest is the wave propagation in the directions perpendicular ($\xi=0$) and parallel ($\eta=0$) to the layering. In these directions the second solution degenerates because according to (15) $\omega_2(0,\eta)=0$ and $\omega_2(\xi,0)=0$. It means that the waves corresponding to the second solution do not exist in these directions. For the first solution the frequency according to (15) is

$$\sqrt{\rho}\,\omega_1(0,\eta) = \sqrt{\frac{1-\nu}{1-2\nu}}\eta,\quad \sqrt{\rho}\,\omega_1(\xi,0) = \sqrt{\frac{1-\nu}{1-2\nu}}\xi\cdot \tag{22}$$

and, subsequently, the components of the phase velocity become

$$\left(v_{px}\right)_1(0,\eta)=0\,,\ \left(v_{px}\right)_1(\xi,0)=\sqrt{\frac{1}{\rho}\frac{1-\nu}{1-2\nu}}\,,\ \left(v_{py}\right)_1(0,\eta)=\sqrt{\frac{1}{\rho}\frac{1-\nu}{1-2\nu}}\,,\ \left(v_{py}\right)_1(\xi,0)=0\,. \tag{23}$$

It follows from (23) that in the direction perpendicular to layering (ξ=0) the horizontal component of the phase velocity is zero and the phase velocity vector is normal to the layering. Analogously, in the direction parallel to layering (η=0) the vertical component of the phase velocity is zero and the phase velocity vector is collinear to the direction of layering.

At this stage, it is important to emphasize that generally the direction of the phase velocity does not coincide with the wave vector. However, in the directions parallel and perpendicular to the layering the phase velocity is collinear to the wave vector due to orthotropic symmetry. It is also important to note that in these directions there is no dispersion relationship since the components of the phase velocity vector v_{px} and v_{py} do not depend on the wave numbers ξ and η (see (23)).

The components of the phase velocity (20), (21) in the polar coordinates (r, φ) read

$$\left(v_{px}\right)_{1,2}=\frac{1-\nu}{1-2\nu}\frac{\cos\varphi}{2\sqrt{\rho}}\frac{\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}\pm\frac{1-\nu}{1-2\nu}-\frac{2\sin^2\varphi}{1-\nu}}{\sqrt{\frac{1-\nu}{2(1-2\nu)}\pm\frac{1}{2}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}}\,, \tag{24}$$

$$\left(v_{py}\right)_{1,2}=\frac{1-\nu}{1-2\nu}\frac{\sin\varphi}{2\sqrt{\rho}}\frac{\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}\pm\frac{1-\nu}{1-2\nu}-\frac{2\cos^2\varphi}{1-\nu}}{\sqrt{\frac{1-\nu}{2(1-2\nu)}\pm\frac{1}{2}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}}\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}}\,. \tag{25}$$

It is clear from (24) and (25) that the components of the phase velocity do not depend on the absolute value of the phase vector $\{\xi,\eta\}$ $r=(\xi^2+\eta^2)^{1/2}$, but only on the direction $\varphi=\arctan(\eta/\xi)$. Thus, in the long wave length approximation, the wave velocity depends on the direction of travel (anisotropy), so there is no dispersion relationship as such.

Figure 1 shows the normalised $v\to v(\rho E^{-1}(1+\nu))^{1/2}$, ρ=1, $E/(1+\nu)$=1 phase velocity for the first and the second solutions in the first quadrant $0<\varphi<\pi/2$, other quadrants being symmetrical.

3. WAVE AMPLITUDES

We consider now the homogeneous system (6)-(8) and investigate the ratio of the amplitudes:

$$\frac{B_3}{B_2}=\frac{G'' i\xi}{B\xi^2+G''-\tilde{J}\omega^2}\,,\qquad \frac{B_1}{B_2}=-\frac{A_{12}\xi\eta}{A_{11}\xi^2-\rho\omega^2}\,. \tag{26}$$

It is seen from (26) that the amplitudes of the longitudinal and shear waves are of the same order of magnitude, while the rotational (responsible for bending) wave is dominated by them. In other words, the displacements caused by the rotational wave are much smaller than the ones caused by the longitudinal and shear waves: $|bB_3/B_2|\ll 1$, $|bB_3/B_1|\ll 1$. Expressions (26) after the insertion of the solution (15) and the normalization (10) have, in the polar coordinates, the form:

$$\left(\frac{B_3}{B_2}\right)_{1,2}=\frac{ir\cos\varphi}{\frac{b^2r^2\cos^2\varphi}{6(1-\nu)}+1-\frac{\tilde{J}r^2}{\rho}\left(\frac{1-\nu}{1-2\nu}\pm\sqrt{\frac{(1-\nu)^2}{(1-2\nu)^2}-\frac{\sin^2 2\varphi}{1-2\nu}}\right)}\,, \tag{27}$$

$$\left(\frac{B_1}{B_2}\right)_{1,2}=-\frac{\dfrac{\nu}{1-2\nu}\cos\varphi\sin\varphi}{\dfrac{1-\nu}{1-2\nu}\cos^2\varphi-\dfrac{1-\nu}{2(1-2\nu)}\mp\dfrac{1}{2}\sqrt{\dfrac{(1-\nu)^2}{(1-2\nu)^2}-\dfrac{\sin^2 2\varphi}{1-2\nu}}}. \tag{28}$$

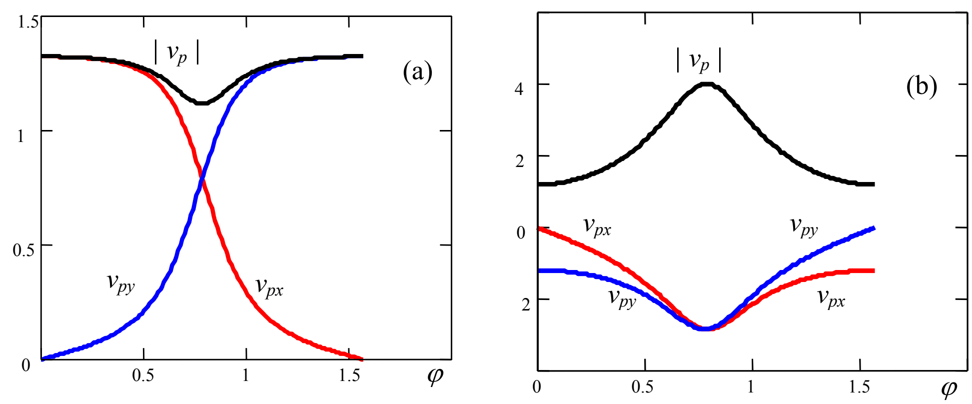

Figure 1: The first (a) and the second (b) solution for the normalized phase velocity (ν=0.3).

Now the directions perpendicular and normal to layering will be considered. The second solution degenerates in these two directions and waves associated with it vanish. The first solution gives the following behaviour for the normalized according to (10) expressions (26):

$$\left(\frac{B_3}{B_2}\right)_1 \sim i\xi+O(\xi^3),\quad \xi\to 0,\qquad \left(\frac{B_1}{B_2}\right)_1 \sim \frac{\nu}{1-\nu}\frac{\xi}{\eta}+O(\xi^3),\quad \xi\to 0, \tag{29}$$

$$\left(\frac{B_3}{B_2}\right)_1 \sim i\xi+O(\eta^2),\quad \eta\to 0,\qquad \left(\frac{B_1}{B_2}\right)_1 \sim \frac{1-\nu}{\nu}\frac{\xi}{\eta}+O(\eta),\quad \eta\to 0. \tag{30}$$

Since the amplitudes of waves must be bounded (otherwise the energy would be infinite), it follows from the (29) that in the direction perpendicular to layering (ξ=0) there is no rotational wave (B_3~ξB_2, $\xi\to 0$, B_2 is bounded $\Rightarrow B_3=0$). There is no shear wave (B_1~$(\xi/\eta)B_2$, $\xi\to 0$, B_2 is bounded $\Rightarrow B_1=0$) in this direction either. It means that only longitudinal waves ($B_2\neq 0$) propagate in the direction normal to layering, while both shear and rotational ones vanish.

In the direction parallel to layering (η=0) from (30) it follows that there is no shear wave (B_2~$(\eta/\xi)B_1$, $\eta\to 0$, B_1 is bounded $\Rightarrow B_2=0$). There is no rotational wave (B_3~ξB_2, $\eta\to 0$, $B_2=0$ $\Rightarrow B_3=0$) in this direction either. It means that only longitudinal waves ($B_1\neq 0$) propagate in the direction parallel to layering, while both shear and rotational ones vanish. These results are summarized in Table1.

Table 1: Wave amplitudes in directions normal and parallel to the layering.

ξ=0	η=0
B_1=0 B_3=0	B_2=0 B_3=0

It is important to conclude that only the longitudinal wave exists in the directions strictly perpendicular and parallel to the layering with both the rotational and shear waves vanishing.

4. A POSSIBLE MECHANISM OF IMPACT FAILURE

Our model offers a partial explanation why a relatively weak, very rapid loading is sometimes sufficient to break even substantial decks of brittle plates.

Since only the longitudinal wave is available in the direction normal to the layering, the impact energy is transmitted from layer to layer in the most efficient way. Therefore, if the impact causes breakage, one can expect that the most possible number of layers will be affected. Had the medium been isotropic and homogeneous both longitudinal and shear waves would have co-existed in the direction perpendicular to the layering. Thus, the impact energy would have been wasted in the shear waves that do not transmit the impact to the adjacent layer. Consequently, a non-layered material would be hard to break the way the karate expert breaks the pile of slabs.

5. CONCLUSION

In a Cosserat continuum that models layered material with sliding layers three types of waves exist: longitudinal, shear and rotational, the letter being associated with the Cosserat rotation. The wave velocity in this approximation depends on the direction of travelling but not on the absolute value of the wave vector, so there is no dispersion relationship as such. Generally, the direction of the phase velocity does not coincide with the wave vector. However, in the directions parallel and perpendicular to the layers the phase velocity is collinear with the wave vector. The magnitude of longitudinal and shear waves dominates that of rotational waves. Only longitudinal waves exist in the directions perpendicular and parallel to the layers with both the rotational and shear waves vanishing. The fact that, in the direction normal to layering, only longitudinal waves can propagate may have a profound effect on the fracture propagation caused by an impact action perpendicular to the layering. Indeed, the impact will be conveyed from one layer to another by the longitudinal wave. When the first layer breaks the longitudinal wave will transmit the load to the next one with the energy dissipation associated with the rotational and shear waves being minor. Thus under the impact the layers will break in succession. This is a possible mechanism by which a karate expert easily breaks a pile of slabs.

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REFERENCES

1. Pasternak, E. *Generalised homogenisation procedures for granular and layered materials*, PhD thesis, The University of Western Australia (2002).
2. Sulem, J. and Mühlhaus, H.-B. A Continuum Model for Periodic Two-dimensional Block Structures, *Mechanics of Cohesive-Frictional Materials*, **2**, 31-46 (1997).
3. Mühlhaus, H.-B. and Oka, F., Dispersion and Wave Propagation in Discrete and Continuous Models for Granular Materials, *Int. J. Solids Structures*, **33**, 2841-2858 (1996).
4. Pasternak, E. and Mühlhaus, H.-B., Cosserat and non-local continuum models for problems of wave propagation in fractured materials, In: *Structural Failure and Plasticity -IMPLAST2000* (X.L. Zhao and R.H. Grzebieta, Eds.), Pergamon, Amsterdam, 741-746 (2000).