NONLOCAL DAMAGE THEORIES BASED ON BALANCES OF MATERIAL FORCES

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ABSTRACT

A thermodynamical consistent weakly nonlocal (gradient-type) theory of brittle and ductile damage is presented. The theory is based on two additional integral balance laws of material forces acting on microcracks, microvoids and dislocations, respectively. Assuming general constitutive equations, it is shown that physical and material forces consist of two parts, a non-dissipative (equilibrium) and a dissipative (non-equilibrium) part. Correspondingly, under specified assumptions the constitutive equations can be expressed by two coupled potentials, the free energy and a dissipation pseudo-potential.

Applying the general theory to isotropic gradient damage coupled with small strain plasticity some engineering structures with brittle and ductile material behavior are analysed numerically.

1 INTRODUCTION

Since Leibfried [1] and Eshelby [2] it is known that there are forces on dislocations and elastic singularities, which are not physical forces. In Naghdi and Srinivarsa [3], Stumpf and Le [4] and Le and Stumpf [5], gradient theories of finite elastoplasticity were derived characterized by an additional balance law of material forces acting on dislocations. Frémond and Nedjar [6] formulated a virtual work principle for isotropic brittle damage leading to the classical force balance and an additional scalar-valued material force balance.

In the classical concept of local elastoplasticity and damage mechanics independent balance laws of material forces are not introduced. Having therefore less equations than unknowns, evolution laws have to be assumed leading to essential drawbacks: (i) there is no information how to chose evolution laws in general, (ii) FE-solution algorithms suffer from strong mesh-dependency, if localization occurs.

The aim of this paper is to derive a thermodynamically consistent weakly nonlocal (gradient-type) damage theory for brittle and ductile continua, which is able to describe appropriately the dissipative process of deformation coupled with damage and plastic evolution. For simplicity we restrict our considerations here to quasi-static, isothermal processes.

2 KINEMATIC AND STRESS VARIABLES

While the deformation of continua in the physical space can be described by a displacement vector and the physical deformation gradient $\mathbf{F}$, respectively. we chose a damage tensor $\mathbf{F}^{\mathrm{d}}$ and a plastic deformation tensor $\mathbf{F}^{\mathrm{p}}$ as independent kinematical variables of the material space. To derive a weakly nonlocal non-equilibrium theory, we chose the ordered set of kinematical variables

$$\boldsymbol{\varepsilon} \equiv \{\mathbf{F}, \mathbf{F}^{\mathrm{d}}, \mathbf{F}^{\mathrm{p}}, \nabla\mathbf{F}^{\mathrm{d}}, \nabla\mathbf{F}^{\mathrm{p}}\} \tag{1}$$

and their rates

$$\dot{\boldsymbol{\varepsilon}} \equiv \{\dot{\mathbf{F}}, \dot{\mathbf{F}}^{\mathrm{d}}, \dot{\mathbf{F}}^{\mathrm{p}}, \nabla\dot{\mathbf{F}}^{\mathrm{d}}, \nabla\dot{\mathbf{F}}^{\mathrm{p}}\} \tag{2}$$

as point of departure.

The ordered set of physical and material stress tensors power-conjugate to (1) and (2), respectively, is denoted by

$$\boldsymbol{\sigma} = \{\mathbf{T}, \mathbf{T}^{\mathrm{d}}, \mathbf{T}^{\mathrm{p}}, \mathbb{H}^{\mathrm{d}}, \mathbb{H}^{\mathrm{p}}\}, \tag{3}$$

where $\mathbf{T}$ is the physical first Piola-Kirchhoff stress tensor and $\mathbf{T}^{\mathrm{d}}, \mathbf{T}^{\mathrm{p}}, \mathbb{H}^{\mathrm{d}}, \mathbb{H}^{\mathrm{p}}$ are material stress tensors of second and third order.

3 GOVERNING EQUATIONS OF DISSIPATIVE PROCESSES

The classical local balance laws of forces and couples in the physical space are

$$\operatorname{Div}\mathbf{T} + \mathbf{b} = \mathbf{0}, \qquad \mathbf{T}\mathbf{F}^{\mathrm{T}} - \mathbf{F}\mathbf{T}^{\mathrm{T}} = \mathbf{0}, \tag{4}$$

where $\mathbf{b}$ is the volume force vector.

Additionally, we postulate global balances of the material forces acting on microdefects and dislocations. By localization we obtain their local forms,

$$\begin{aligned} &\operatorname{Div}\mathbb{H}^{\mathrm{d}} - \mathbf{T}^{\mathrm{d}} + \mathbf{G}^{\mathrm{d}} = \mathbf{0}, \\ &\operatorname{Div}\mathbb{H}^{\mathrm{p}} - \mathbf{T}^{\mathrm{p}} + \mathbf{G}^{\mathrm{p}} = \mathbf{0}, \end{aligned} \tag{5}$$

where $\mathbf{G}^{\mathrm{d}}$, $\mathbf{G}^{\mathrm{p}}$ are external influences as chemical reactions breaking internal material bonds.

From the global form of the second law of thermodynamics for physical and material space the following local form of the dissipation inequality can be derived

$$d \equiv \mathbf{T} \bullet \dot{\mathbf{F}} + \mathbf{T}^{\mathrm{d}} \bullet \dot{\mathbf{F}}^{\mathrm{d}} + \mathbb{H}^{\mathrm{d}} \bullet \nabla\dot{\mathbf{F}}^{\mathrm{d}} + \mathbf{T}^{\mathrm{p}} \bullet \dot{\mathbf{F}}^{\mathrm{p}} + \mathbb{H}^{\mathrm{p}} \bullet \nabla\dot{\mathbf{F}}^{\mathrm{p}} - \dot{\psi} \geq 0, \tag{6}$$

where ψ is the free energy.

4 CONSTITUTIVE AND GOVERNING EQUATIONS

The constitutive equations are assumed in general form,

$$\psi = \hat{\psi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}), \qquad \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}). \tag{7}$$

Introducing (7) into the second law (6) leads to the thermodynamically admissible form of the constitutive equations

$$\psi = \hat{\psi}(\boldsymbol{\varepsilon}), \qquad \boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) = \partial_{\boldsymbol{\varepsilon}}\hat{\psi}(\boldsymbol{\varepsilon}) + \hat{\boldsymbol{\sigma}}_*(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}). \tag{8}$$

It follows that the free energy cannot depend on the rates $\dot{\boldsymbol{\varepsilon}}$ and that the physical and material stresses consists of two parts, one part which can be derived from the free energy, and one part which depends also on $\dot{\boldsymbol{\varepsilon}}$. This part denoted by an asterisk can be considered as the driving forces on defects.

Introducing (8) into the dissipation inequality (6) leads to the entropy production of the dissipative driving forces on defects,

$$d = \hat{\boldsymbol{\sigma}}_*(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \bullet \dot{\boldsymbol{\varepsilon}} \geq 0. \tag{9}$$

In general (9) need not be integrable. However, if its is integrable then there exists a scalar-valued function φ,

$$\varphi = \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}), \qquad \hat{\varphi}(\boldsymbol{\varepsilon}, \mathbf{0}) = 0, \tag{10}$$

called the dissipation pseudo-potential, from which the dissipative driving stresses can be determined

$$\hat{\boldsymbol{\sigma}}_*(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) = \partial_{\dot{\boldsymbol{\varepsilon}}}\hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}). \tag{11}$$

From (11) and (8) the thermodynamcilly admissible constitutive equations are obtained in the form

$$\boldsymbol{\sigma} = \hat{\boldsymbol{\sigma}}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) = \partial_{\boldsymbol{\varepsilon}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\dot{\boldsymbol{\varepsilon}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}), \tag{12}$$

and with (11) the dissipaton inequality (9) reads

$$\mathscr{d} = \partial_{\dot{\boldsymbol{\varepsilon}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \bullet \dot{\boldsymbol{\varepsilon}} \geq 0. \tag{13}$$

Finally, introducing (12) into the physical and material balance laws $(4)_1$ and (5) we obtain the governing equations of the dissipative process of deformation of structures coupled with damage and plastic evolution

$$\begin{aligned} &\mathrm{Div}\left(\partial_{\mathbf{F}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\dot{\mathbf{F}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})\right) + \mathbf{b} = \mathbf{0}, \\ &\mathrm{Div}\left(\partial_{\nabla \mathbf{F}^{\mathrm{d}}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\nabla \dot{\mathbf{F}}^{\mathrm{d}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})\right) - \left(\partial_{\mathbf{F}^{\mathrm{d}}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\dot{\mathbf{F}}^{\mathrm{d}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})\right) + \mathbf{G}^{\mathrm{d}} = \mathbf{0}, \\ &\mathrm{Div}\left(\partial_{\nabla \mathbf{F}^{\mathrm{p}}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\nabla \dot{\mathbf{F}}^{\mathrm{p}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})\right) - \left(\partial_{\mathbf{F}^{\mathrm{p}}} \hat{\psi}(\boldsymbol{\varepsilon}) + \partial_{\dot{\mathbf{F}}^{\mathrm{p}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}})\right) + \mathbf{G}^{\mathrm{p}} = \mathbf{0}. \end{aligned} \tag{14}$$

Equations $(14)_1$ and $(14)_2$ represent the governing equations of thermodynamically admissible gradient theory of finite elastoplasticity. Assuming additionally $\dot{\mathbf{F}} = \mathbf{0}$, neglecting all gradient terms and introducing the multiplicative decomposition formula, $\mathbf{F} = \mathbf{F}^{\mathrm{e}}\mathbf{F}^{\mathrm{p}}$ with the elastic deformation $\mathbf{F}^{\mathrm{e}}$, the local theory of finite elastoplasticity of Cermelli et al. [7] with nine independent components of $\dot{\mathbf{F}}^{\mathrm{p}}$ is obtained.

4 LIFE-TIME ANALYSIS OF STRUCTURES

Integration of the dissipation (13) as a function of the position vector $\mathbf{X}$ in the underformed and homogeneous reference configuration and time t over the material body leads to

$$\mathscr{D}(\mathrm{t}) = \int_{\mathrm{P}} \partial_{\dot{\boldsymbol{\varepsilon}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \bullet \dot{\boldsymbol{\varepsilon}} \mathrm{dv}. \tag{15}$$

Integration of the dissipative process in the time interval $[0, \mathrm{T}]$ yields the dissipated energy

$$\mathscr{D}(\mathrm{T}) = \int_0^{\mathrm{T}} \left(\int_{\mathrm{P}} \partial_{\dot{\boldsymbol{\varepsilon}}} \hat{\varphi}(\boldsymbol{\varepsilon}, \dot{\boldsymbol{\varepsilon}}) \bullet \dot{\boldsymbol{\varepsilon}} \mathrm{dv} \right) \mathrm{dt}, \tag{16}$$

where $\boldsymbol{\varepsilon}$ can be considered as parameter.

In Fig. 1 the hypersurface of the free energy $\hat{\psi}(\boldsymbol{\varepsilon})$ with $\mathscr{d}(\boldsymbol{\varepsilon}, \mathbf{0}) = 0$ and the hypersurface of dissipation $\mathscr{D}(\mathrm{T})$ are represented graphically with an assumed process-path of the deformation coupled with plastic and damage evolution as function of time:

- At time t_0 the material body is in a homogeneous and undeformed reference state.
- In time interval $[\mathrm{t}_0, \mathrm{t}^{\mathrm{p}}]$ the material body is deformed elastically.
- At time t^{p} the yield point is reached and plastic flow with increasing dissipation beginns.
- During the time interval $[\mathrm{t}^{\mathrm{p}}, \mathrm{t}^{\mathrm{d}}]$ there is elastic-plastic deformation.
- At time t^{d} the threshold value is reached and the process of damage with increasing dissipation beginns.
- At any time instant T the total dissipation of the material body is defined by eqn (16).

The further behavior of the process-path is crucial for the life-time analysis of engineering structures. If there is a continuous increase of the dissipated energy the structure will be destroyed. If the process-path reaches the free energy hypersurface and all further deformations will be purely elastic, then there is shakedown of the structure. This is essentially important for cyclic deformations (see e.g. Stumpf [8], Weichert and Hachemi [9], Druganov and Roman [10]).

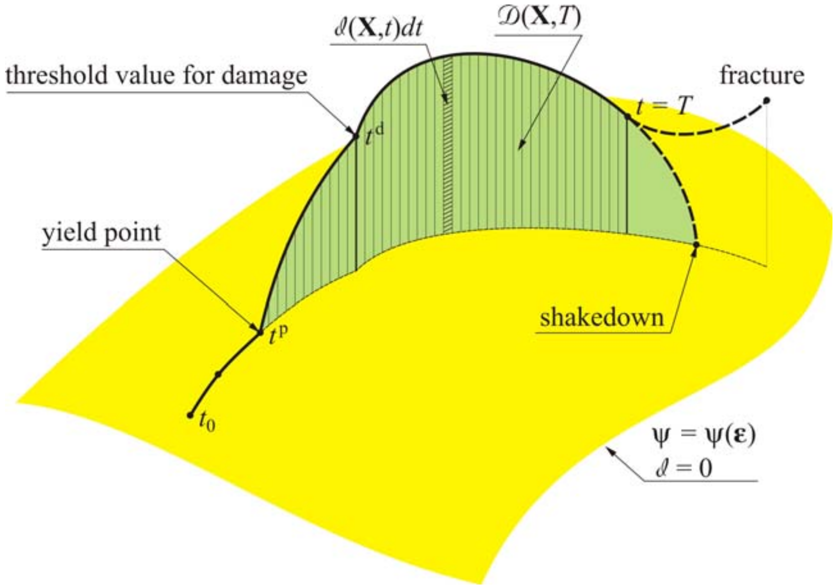

Figure 1: Graphic illustration of dissipative processes.

5 SPECIAL THEORIES AND EXAMPLES

The damage concept presented in this paper leads to a system of partial differential equations (14) with respect to the displacement field $\mathbf{u}$ with associated deformation gradient $\mathbf{F} = \mathbf{1} + \nabla\mathbf{u}$, which is coupled with a system of partial differential equations in the damage tensor $\mathbf{F}^{\mathrm{d}}$ and the plastic deformation tensor $\mathbf{F}^{\mathrm{p}}$. These coupled systems of differential equations must be solved for $\mathbf{u}$, $\mathbf{F}^{\mathrm{d}}$ and $\mathbf{F}^{\mathrm{p}}$ for the chosen constitutive equations and prescribed boundary and initial conditions.

From the computational point of view, the solution of the formulated initial-boundary value problem typically involves a finite-dimensional approximation of the continuous (infinite-dimensional) problem and the numerical integration of the resulting semi-discrete problem. The derivation of FE-solution algorithms relies crucially on the weak form of the balance laws for physical and material forces which states that

$$\begin{aligned}&\int_{\mathrm{B}}\left(\mathbf{T}\bullet\delta\mathbf{F}+\mathbf{T}^{\mathrm{d}}\bullet\delta\mathbf{F}^{\mathrm{d}}+\mathbb{H}^{\mathrm{d}}\bullet\nabla\delta\mathbf{F}^{\mathrm{d}}+\mathbf{T}^{\mathrm{p}}\bullet\delta\mathbf{F}^{\mathrm{p}}+\mathbb{H}^{\mathrm{p}}\bullet\nabla\delta\mathbf{F}^{\mathrm{p}}\right)\mathrm{dv}\\-&\int_{\mathrm{B}}\left(\mathbf{b}\bullet\delta\mathbf{u}+\mathbf{G}^{\mathrm{d}}\bullet\delta\mathbf{F}^{\mathrm{d}}+\mathbf{G}^{\mathrm{p}}\bullet\delta\mathbf{F}^{\mathrm{p}}\right)\mathrm{dv}-\int_{\partial\mathrm{B}}\left(\mathbf{Tn}\bullet\delta\mathbf{u}+\mathbb{H}^{\mathrm{d}}\mathbf{n}\bullet\delta\mathbf{F}^{\mathrm{d}}+\mathbb{H}^{\mathrm{p}}\mathbf{n}\bullet\delta\mathbf{F}^{\mathrm{p}}\right)\mathrm{da}=0\end{aligned}\tag{17}$$

for all virtual displacements $\delta\mathbf{u}$, virtual damage tensors $\delta\mathbf{F}^{\mathrm{d}}$ and virtual plastic tensors $\delta\mathbf{F}^{\mathrm{p}}$.

Further simplifications of the general theory are obtained, if the constitutive equations are assumed in specific form. In the case of isotropic damage of engineering structures the damage tensor reduces to $\mathbf{F}^{\mathrm{d}} = \mathrm{D}\mathbf{1}$, where $\mathrm{D}(\mathbf{X},\mathrm{t})$ is the isotropic damage function, $0 \le \mathrm{D} \le 1$, and $\mathbf{1}$ is the identity tensor. With $\dot{\mathbf{F}} = \mathbf{0}$, $\mathbf{G}^{\mathrm{d}} = \mathbf{0}$ and assuming $\nabla\dot{\mathrm{D}} = 0$, we obtain from $(14)_1$ and $(14)_2$ the governing equations of the gradient theory of isotropic damage,

$$\begin{aligned}\mathrm{Div}\left(\mathbf{F}\partial_{\mathbf{E}}\hat{\psi}(\mathbf{E},\mathbf{E}^{\mathrm{p}},\mathrm{D},\nabla\mathrm{D})\right)+\mathbf{b}=\mathbf{0},\\ \mathrm{Div}\left(\partial_{\nabla\mathrm{D}}\hat{\psi}(\mathbf{E},\mathbf{E}^{\mathrm{p}},\mathrm{D},\nabla\mathrm{D})\right)-\left(\partial_{\mathrm{D}}\hat{\psi}(\mathbf{E},\mathbf{E}^{\mathrm{p}},\mathrm{D},\nabla\mathrm{D})+\partial_{\dot{\mathrm{D}}}\hat{\varphi}(\mathbf{E},\mathbf{E}^{\mathrm{p}},\mathrm{D},\nabla\mathrm{D},\dot{\mathrm{D}})\right)=\mathbf{0},\end{aligned}\tag{18}$$

where we replaced $\mathbf{F}$ and $\mathbf{F}^p$ by their objective forms, the Green strain tensor $\mathbf{E}$ and the plastic Green strain tensor $\mathbf{E}^p$.

In the case of ductile material behavior we can combine (18) with the classical model of local small strain elastoplasticity with the additive strain decomposition

$$\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p , \tag{19}$$

where $\mathbf{E}^e$ is the elastic strain tensor.

If we assume that the free energy depends only on the elastic deformation, then it takes the simple form

$$\psi = \frac{1}{2}(1-\mathrm{D})\hat{\psi}(\mathbf{E}-\mathbf{E}^p) + \frac{1}{2}\mathrm{k}\nabla\mathrm{D}\bullet\nabla\mathrm{D}. \tag{20}$$

With $\nabla\mathrm{D} = 0$, $\nabla\dot{\mathrm{D}} = 0$, we chose the dissipation pseudo-potential (10) in the form

$$\varphi = \hat{\varphi}(\mathbf{E}^e, \mathrm{D}, \dot{\mathrm{D}}, \dot{\mathbf{E}}^p), \tag{21}$$

which allows an additive split into a damage and plastic part

$$\varphi = \hat{\varphi}^d(\mathbf{E}^e, \mathrm{D}, \dot{\mathrm{D}}) + \hat{\varphi}^p(\mathrm{D}, \dot{\mathbf{E}}^p), \tag{22}$$

(see also Nedjar [11] and the material parameters given there).

A straightforward way to construct a FE-approximation is the use of a "displacement" based finite element method associated with the principle of virtual work (17). In this respect it is important to note that although the considered theory is weakly nonlocal (gradient-type), the presented formulation based on the balance laws for physical and material forces allows to use standard C^0 finite element approximations. Moreover, with the restriction to plane problems, the simplest finite element formulation may be based on triangular elements having three nodes and linear interpolations of the displacement vector and the damage variable. This kind of finite element is used in the numerical analysis of the splitting test shown in Figure 2.

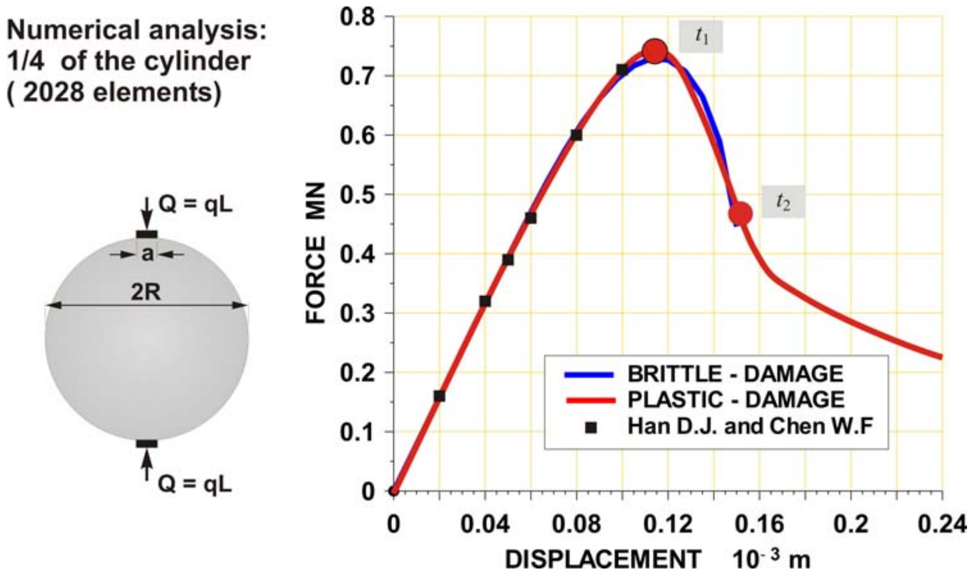

Figure 2: Splitting tests - comparison of brittle and ductile damage solutions.

In Figure 2, on the right side, the deformations in the physical space at various time instants are presented for brittle and ductile material behavior together with results of Han and Chen [12]. Figure 3 shows the damage and plastic evolution in the material space at time instants t_1 and t_2 .

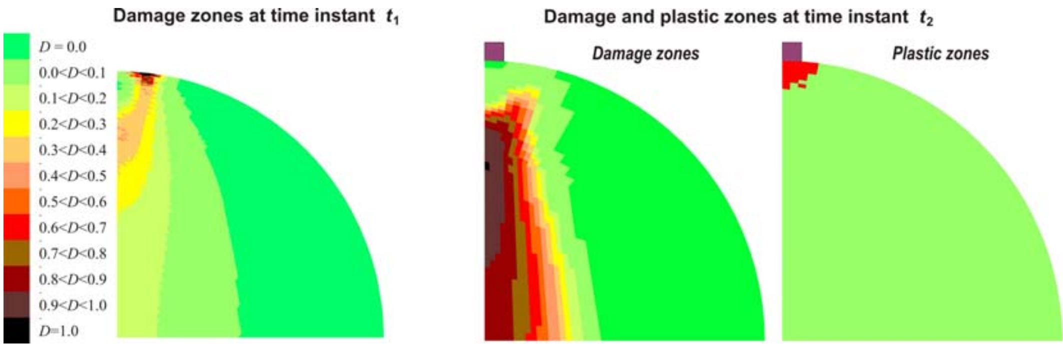

Figure 3: Damage and plastic zones at specific levels of deformation.

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