EVALUATION OF STRESS INTENSITY FACTORS AND T-STRESS IN FUNCTIONALLY GRADED MATERIALS USING THE INTERACTION INTEGRAL METHOD

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ABSTRACT

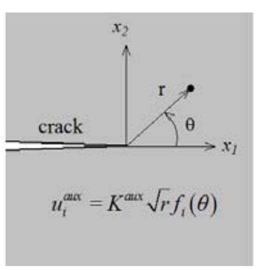
The interaction integral method is revisited to evaluate both the mixed-mode stress intensity factors and the T-stress in functionally graded materials under mechanical loading. In particular, a non-equilibrium formulation is developed in an equivalent domain integral form of the interaction integral, which is naturally suitable to the finite element method. The material gradation is integrated into the element stiffness matrix using the generalized isoparametric formulation. The types of material gradation include continuum functions such as an exponential function, but the present formulation can be readily extended to micromechanical models. This paper presents a fracture problem with an inclined crack and assesses the accuracy of the present method compared with available semi-analytical solutions. The revisited interaction integral method is shown to be an accurate scheme for evaluating such parameters in functionally graded materials.

1 INTRODUCTION

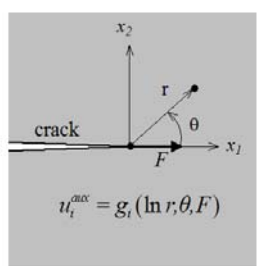
Mixed-mode fracture of functionally graded materials (FGMs) has been investigated by evaluating mixed-mode stress intensity factors (SIFs) [1-3] and the T-stress [4]. Recently, the interaction integral method has been used to evaluate the mixed-mode SIFs [5-9] and T-stress [9-10] in FGMs. This paper aims at the development of the non-equilibrium formulation to evaluate the mixed-mode SIFs and the T-stress in FGMs and the assessment of the accuracy of the method.

2 AUXILIARY FIELDS

The interaction integral uses auxiliary fields, such as stresses, strains and displacements [11]. Here we adopt fields (displacements and strains) originally obtained for homogeneous materials and use a formulation accounting for non-equilibrium of the auxiliary stresses [9,10]. For the mixed-mode SIFs, we select the auxiliary displacement and strain fields as the Williams's [12] crack-tip asymptotic fields with the crack-tip material properties (see Figure 1(a)). For the T-stress, we choose such fields as those [13] due to a point force in the x_1 direction, applied to the tip of a semi-infinite crack in an infinite homogeneous body as shown in Figure 1(b).



(a)



(b)

Figure 1: Auxiliary fields: (a) Williams's solution [12] for the SIF; (b) Michell's solution [13] for the T-stress.

3 THE INTERACTION INTEGRAL

The interaction integral (M-integral) is derived from the path-independent J-integral [14] for two admissible states of a cracked elastic FGM. The standard J-integral is given by [14]

$$J = \lim_{\Gamma_s \to 0} \int_{\Gamma_s} (W\delta_{1j} - \sigma_{ij} u_{i,1}) n_j \; d\Gamma, \tag{1}$$

where W is the strain energy density, σ_{ij} denotes the stress, u_i denotes the displacements, and n_j is the outward normal vector to the contour Γ, as shown in Figure 2.

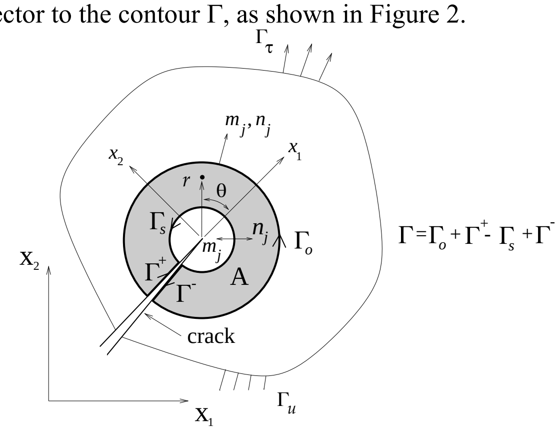


Figure 2: Conversion of the contour integral into EDI.

Using the divergence theorem and the weight function q (varying from unity at the crack tip to zero on Γ_0), one obtains the following EDI [15]

$$J = \int_A (\sigma_{ij} u_{i,1} - W\delta_{1j}) q_{,j} \; dA + \int_A (\sigma_{ij} u_{i,1} - W\delta_{1j})_{,j} q \; dA \tag{2}$$

The J-integral of the superimposed fields (actual and auxiliary) is conveniently decomposed into

$$J^s = J + J^{aux} + M, \tag{3}$$

where the resulting form of the M-integral considering non-equilibrium formulation is given by

$$M = \int_A (\sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j}) q_{,j} \; dA + \int_A (\underline{\sigma_{ij,j}^{aux} u_{i,1}} - C_{ijkl,1} \varepsilon_{kl} \varepsilon_{ij}^{aux}) q \; dA \tag{4}$$

where the underlined term is a non-equilibrium term that appears due to non-equilibrium of auxiliary stress fields, which must be considered to obtain converged solutions. The existence of the resulting M-integral in eqn (4) as the limit $r \to 0$ has been proved in the references [9,10]. For numerical computation by means of the FEM, the M-integral is evaluated first in the global coordinates (M_{global}) and then transformed to the local coordinates (M_{local}=M). The global interaction integral (M_{global}) is obtained as (m=1, 2) [9,10]

$$\left(M_m\right)_{global} = \int_A (\sigma_{ij} u_{i,m}^{aux} + \sigma_{ij}^{aux} u_{i,m} - \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{mj}) \frac{\partial q}{\partial X_j} \; dA + \int_A (\sigma_{ij,j}^{aux} u_{i,m} - C_{ijkl,m} \varepsilon_{kl} \varepsilon_{ij}^{aux}) q \; dA \tag{5}$$

where X_1 and X_2 are the global coordinates shown in Figure 2. Now one obtains M_{local} as

$$M_{local} = \left(M_1\right)_{local} = \left(M_1\right)_{global} \cos\theta + \left(M_2\right)_{global} \sin\theta. \tag{6}$$

4 EVALUATION OF THE SIFS

From the relationship between superimposed J-integral and the mode I and mode II SIFs, one obtains the following relationship between M-integral and SIFs:

$$M = 2(K_I K_I^{aux} + K_{II} K_{II}^{aux}) / E_{tip}^* \tag{7}$$

The mode *I* and mode *II* SIFs are decoupled and are evaluated as follows:

$$\begin{aligned} K_I &= M^{(1)} E_{tip}^* / 2, \ (K_I^{aux} = 1.0, K_{II}^{aux} = 0.0), \\ K_{II} &= M^{(2)} E_{tip}^* / 2, \ (K_I^{aux} = 0.0, K_{II}^{aux} = 1.0). \end{aligned} \tag{8}$$

The relationships of eqn (8) are essentially the same as those for homogeneous materials [11] except that, for FGMs, the material properties are evaluated at the crack-tip location [1,2].

5 EVALUATION OF THE T-STRESS

The T-stress (non-singular stress) can be extracted from the interaction integral taking the limit $r \to 0$ of the domain *A* shown in Figure 2. By doing so, one obtains [9,10]

$$M = \lim_{\Gamma_s \to 0} \int_{\Gamma_s} \left\{ \sigma_{ik} \varepsilon_{ik}^{aux} \delta_{1j} - \left(\sigma_{ij} u_{i,1}^{aux} + \sigma_{ij}^{aux} u_{i,1} \right) \right\} n_j \, d\Gamma. \tag{9}$$

Substituting the actual stress fields into eqn (9) and using the following equilibrium

$$F = \lim_{\Gamma_s \to 0} \int_{\Gamma_s} \sigma_{ij}^{aux} n_j \, d\Gamma, \tag{10}$$

one obtains $T = E_{tip}^* M / F$ [9,10].

6 EXAMPLE: INCLINED CENTER CRACK IN A PLATE

Figure 3(a) shows an inclined center crack of length *2a* located with geometric angle θ in an FGM plate under fixed-grip loading, Figure 3(b) shows the complete mesh configuration, and Figure 3(c) shows a mesh detail using 12 sectors (S12) and 4 rings (R4) of elements around crack tips. The mesh discretization consists of 1641 Q8 (eight-node quadrilateral), 94 T6 (regular six-node triangles), and 24 T6qp (singular quarter-point six-node triangles) elements, with a total of 1759 elements and 5336 nodes.

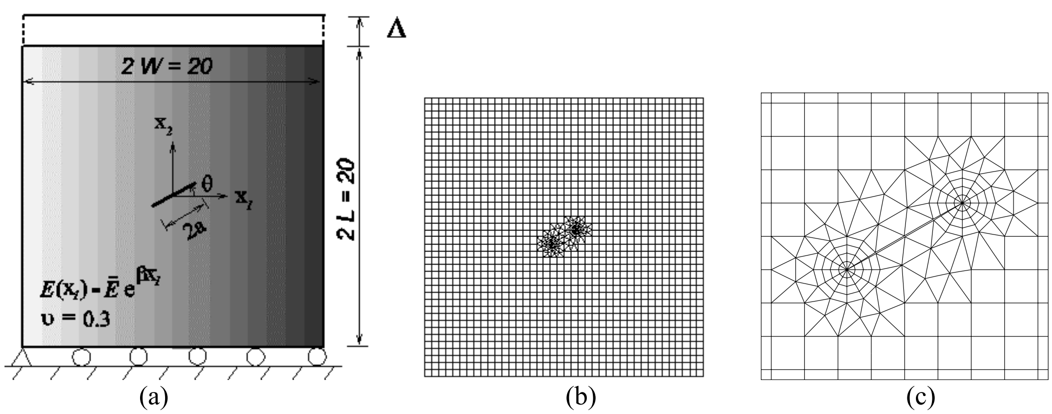


Figure 3: Plate with an inclined crack: (a) geometry and boundary conditions (BCs); (b) complete finite element mesh; (c) mesh detail using 12 sectors and 4 rings around the crack tips (θ = 30°).

The applied load corresponds to $\sigma_{22}\left(X_1,10\right)=\bar{\varepsilon}\,\bar{E}\,e^{\beta X_1}$. The following data were used for the FEM analysis: *a/W*=0.1; *L/W*=1.0; θ=0° to 90°; plane stress; $E\left(X_1\right)=\bar{E}e^{\beta X_1}$; $\bar{E}=1.0$; βa=(0.0,0.5); ν=0.3. Table 1 compares FEM results for normalized SIFs obtained by the interaction integral with those obtained by Konda and Erdogan [16]. Notice that two results are in good agreement (maximum difference 2.1%, average difference 0.7%). Table 2 compares FEM results for T-stress with those obtained by Paulino and Dong [17]. Notice again that two results are in good agreement (maximum difference 1.8%, average difference 1.0% for the homogeneous case with βa =0.0; maximum difference 2.4%, average difference 1.2% for the FGM case with βa=0.5). For a homogeneous material, the FEM results for T-stress for the right crack-tip are the same as those for the left crack-tip. Notice that as the dimensionless material nonhomogeneity parameter βa increases, the T-stress for the right crack-tip *T*(+a) increases within the range of 0° ≤ θ < 90°, however, the T-stress for the left crack-tip *T*(-a) increases in the range of 0° ≤ θ < 45° and then decreases in the range of 45° < θ < 90°.

Table 1: Comparison of normalized SIFs obtained by the interaction integral with those obtained by Konda and Erdogan [16] ($K_0=\bar{\varepsilon}\,\bar{E}\,\sqrt{\pi a}$).

Method	θ (deg)	K_I^+/K_0	K_{II}^+/K_0	K_I^-/K_0	K_{II}^-/K_0
Konda & Erdogan [16]	0	1.424	0.000	0.674	0.000
	18	1.285	0.344	0.617	0.213
	36	0.925	0.548	0.460	0.365
	54	0.490	0.532	0.247	0.397
	72	0.146	0.314	0.059	0.269
	90	0.000	0.000	0.000	0.000
Present	0	1.423	0.000	0.665	0.000
	18	1.283	0.346	0.610	0.210
	36	0.922	0.551	0.455	0.362
	54	0.488	0.534	0.245	0.393
	72	0.145	0.314	0.057	0.267
	90	0.000	0.000	0.000	0.000

Table 2: Comparison of the T-stress obtained by the interaction integral with those obtained by Paulino and Dong [17].

Method	θ (deg)	βa=0.0		βa=0.5	
		T(+a)	T(-a)	T(+a)	T(-a)
Paulino and Dong [17]	0	-0.9999	-0.9999	-0.8670	-0.8766
	15	-0.8660	-0.8660	-0.7483	-0.7631
	30	-0.5001	-0.5001	-0.4200	-0.4444
	45	0.0002	0.0000	0.0393	0.0109
	60	0.4999	0.5000	0.5132	0.4905
	75	0.8660	0.8660	0.8701	0.8585
	90	1.0000	1.0000	1.0000	1.0000
Present	0	-0.9828	-0.9828	-0.8963	-0.8589
	15	-0.8534	-0.8534	-0.7734	-0.7478
	30	-0.4974	-0.4974	-0.4334	-0.4360
	45	-0.0055	-0.0055	0.0361	0.0115
	60	0.4912	0.4912	0.5133	0.4845
	75	0.8592	0.8592	0.8685	0.8502
	90	0.9950	0.9950	0.9945	0.9945

7 CONCLUSIONS

This paper presents an accurate and robust scheme for evaluating the mixed-mode SIFs and T-stress by means of the interaction integral (M-integral) method considering arbitrarily oriented cracks in two-dimensional (2D) elastic FGMs. From the numerical example investigated, we observe that the interaction integral method is accurate in calculating the SIFs and the T-stress in FGMs. Moreover, the material nonhomogeneity βa shows significant influence on the fracture parameters (SIFs and T-stress) in FGMs.

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